

Second Saturday School Solutions

Question 1 [2007 AB1]

$$\text{Area} = \int_{-3}^3 \left[\frac{20}{1+x^2} - 2 \right] dx \approx 37.961 \text{ or } 37.962$$

[R rotated about x-axis]

$$\text{Volume} = \pi \int_{-3}^3 \left[\left(\frac{20}{1+x^2} \right)^2 - (2)^2 \right] dx \approx 1871.190$$

[R with semi-circles perpendicular to x-axis]

$$\text{Volume} = \frac{\pi}{2} \int_{-3}^3 \left[\frac{1}{2} \left(\frac{20}{1+x^2} - 2 \right) \right]^2 dx \approx 174.268$$

be mindful of your grouping symbols

$$r = \frac{1}{2} \left[\frac{20}{1+x^2} - 2 \right]$$

Question 2 [2008AB2]

$$L'(5.5) \approx \frac{L(7) - L(4)}{7 - 4} = \frac{150 - 128}{3} = 8 \frac{\text{peeps}}{\text{hour}}$$

MTAN \approx msec

peeps = people

The average number of people waiting in line during the first 4 hours is approximately

$$\frac{1}{4-0} \int_0^4 L(t) dt \quad [\text{We'll use the trapezoid sum but beware of the unequal intervals!]$$

$$\frac{1}{4} \left[\frac{L(1)+L(0)}{2} (1) + \frac{L(3)+L(1)}{2} (2) + \frac{L(4)+L(3)}{2} (1) \right] = 155.25 \text{ people}$$

TRAP I TRAP II TRAP III

Part (c) is a bit of a drag. We need to use TWO theorems!!

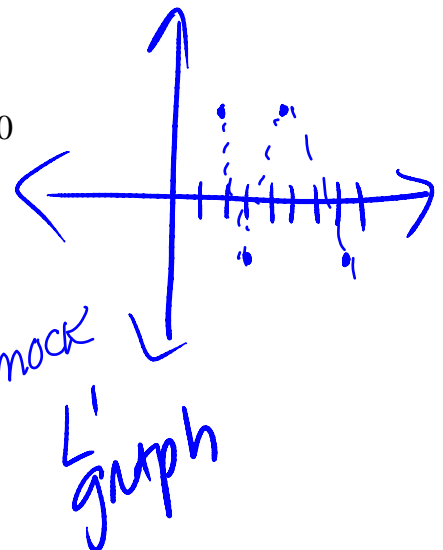
L is differentiable on $[0, 9]$ so the Mean Value Theorem applies. Here goes:

By the MVT, there is a c , $1 < c < 3$, such that $L'(c) = \frac{L(3) - L(1)}{3 - 1} > 0$

By the MVT there is a d , $4 < d < 7$, such that $L'(d) = \frac{L(7) - L(4)}{7 - 4} > 0$

By the MVT there is an r , $3 < r < 4$, such that $L'(r) = \frac{L(4) - L(3)}{4 - 3} < 0$

By the MVT there is an s , $7 < s < 8$, such that $L'(s) = \frac{L(8) - L(7)}{8 - 7} < 0$



Since $L'(t)$ must be continuous, then the Intermediate Value Theorem implies that $L'(t) = 0$ at least three times during the interval $[0, 9]$

Note: You must use Calculus to prove this. You can not simply say that $L(t)$ increases, then decreases, then increases, then decreases.

$r(t)$ is A Rate

$$\int_0^3 r(t) dt \approx 972.784 \text{ Hence, there were approximately 973 tickets sold by 3 p.m.}$$

Another note: Notice that the table went from time = 0 to time = 9 but parts a, b, and d only used parts of the table. Be vigilant!

Leaky Oil Pipeline Problem [2008AB3]

Ah! Related Rates!

Given: $r = 100\text{cm}$, $h = 0.5\text{cm}$, $\frac{dr}{dt} = 2.5 \frac{\text{cm}}{\text{min}}$, $\frac{dV}{dt} = 2000 \frac{\text{cm}^3}{\text{min}}$

$$V = \pi r^2 h$$

$$\frac{d}{dt} V = \frac{d}{dt} (\pi r^2 h)$$

OUR FRIEND, THE PRODUCT RULE

$$\frac{dV}{dt} = \pi \left[2r h \frac{dr}{dt} + r^2 \frac{dh}{dt} \right]$$

$$2000 \frac{\text{cm}^3}{\text{min}} = \pi \left[2(100\text{cm})(0.5\text{cm}) \left(2.5 \frac{\text{cm}}{\text{min}} \right) + (100\text{cm})^2 \frac{dh}{dt} \right]$$

$$\frac{dh}{dt} \approx \approx 0.038 \text{ or } 0.039 \frac{\text{cm}}{\text{min}}$$

$$\frac{dV}{dt} = 2000 - R(t)$$

↑
RATE OF
OIL
being
spilled

↑
RATE OF OIL
being
removed

Let $\frac{dV}{dt} = 0$

So, $0 = 2000 - R(t)$

This occurs when $t = 25$ minutes

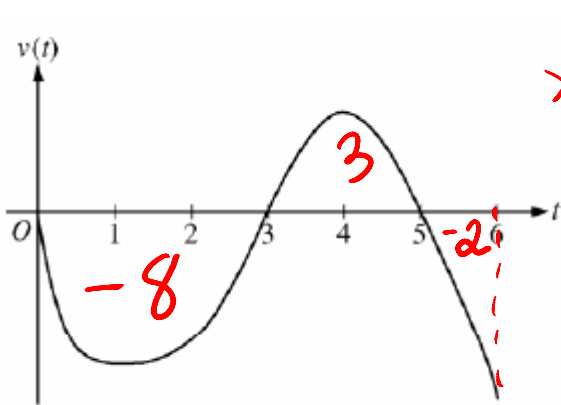
Now analyze: On the time interval $(0, 25)$ $\frac{dV}{dt} > 0$, and for $t > 25$, $\frac{dV}{dt} < 0$

Hence, the oil slick reached its maximum volume 25 minutes AFTER THE REMOVAL DEVICE BEGINS WORKING.

Volume of oil at $t = 25$ minutes

$$V = 60000 + \int_0^{25} [2000 - R(t)] dt \quad \text{This is like the Sandy Point Beach problem.}$$

Notice that the integrand $[2000 - R(t)]$ represents the rate of change of the volume. In other words, $\frac{dV}{dt}$ or $V'(t)$



Graph of v

$$x(t) = \text{POSITION}$$

$$x(0) = -2$$

At $t = 3$, $v(t) = x'(t)$ changes from negative to positive values. [This is at the very least, a relative minimum]

To find the absolute minimum, or where the particle is farthest to the left, we must find $x(0)$, $x(3)$, and $x(6)$

$$x(0) = -2 \quad [\text{This was given.}]$$

$$x(3) = x(0) + \int_0^3 v(t) dt = -2 + -8 = -10$$

Remember: initial position plus the displacement

$$x(6) = x(0) + \int_0^6 v(t) dt = -2 + -8 + 3 + -2 = -9$$

Hence, the particle is farthest left at $t = 3$ and its position is $x(3) = -10$

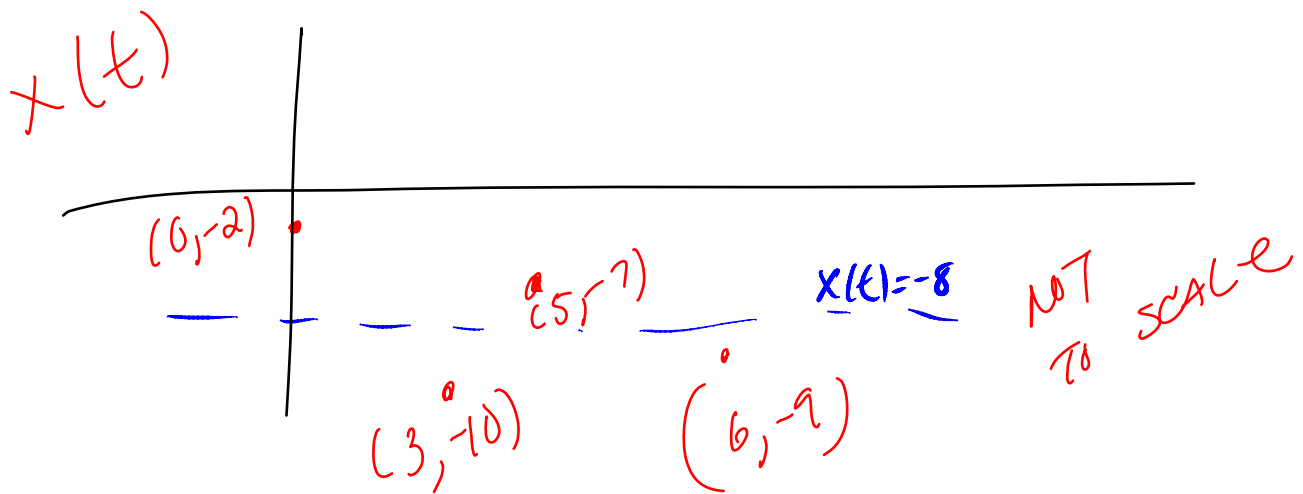
On the time interval $0 \leq t \leq 3$, the particle moves continuously and always to the left. Its position changes from $x(0) = -2$ to $x(3) = -10$.

On the time interval $3 < t \leq 5$, the particle moves continuously and always to the right. Its position changes from $x(3) = -10$ to $x(5) = -7$.

On the time interval $5 < t \leq 6$, the particle moves continuously and always to the left. Its position changes from $x(5) = -7$ to $x(6) = -9$

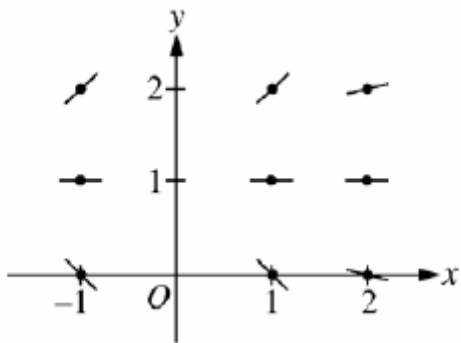
Hence, thus, therefore, by the Intermediate Value Theorem, there are three values of t for which the position, $x(t) = -8$

Here is a graph of the values of the position



On the interval $2 < t < 3$, $v(t) < 0$ AND the graph of $v(t)$ is increasing which means that $a(t) = v'(t) > 0$. Hence, the speed is decreasing on this interval.

On the intervals $0 < t < 1$ and $4 < t < 6$, the graph of $v(t)$ is decreasing. Hence, the acceleration is negative on these intervals.



$$\frac{dy}{dx} = \frac{y-1}{x^2}$$

$$\frac{1}{y-1} dy = x^{-2} dx \quad \text{SEPARATE}$$

$$\int \frac{1}{y-1} dy = \int x^{-2} dx \quad \text{INTEGRATE}$$

$$\ln|y-1| = -\frac{1}{x} + C \quad \text{NOW SOLVE FOR C}$$

$$\ln|0-1| = \frac{-1}{2} + C \quad \text{HENCE, } C = \frac{1}{2}$$

NOW WE NEED TO BE MINDFUL OF THE ABSOLUTE VALUE WHEN SOLVING FOR Y [Did we need those absolute values when solving for C?]

$$e^{\ln|y-1|} = e^{\frac{-1}{x} + \frac{1}{2}}$$

$$|y-1| = e^{\frac{-1}{x} + \frac{1}{2}}$$

$$1-y = e^{\frac{-1}{x} + \frac{1}{2}}$$

$$y = 1 + e^{\frac{-1}{x} + \frac{1}{2}} \quad \text{OR} \quad y = 1 + \left(e^{\frac{1}{2}} \right) \left(e^{\frac{-1}{x}} \right)$$

Remember:

$$|y-1| = \begin{cases} 1-y, & y < 1 \\ y-1, & y \geq 1 \end{cases}$$

our given y
was y=0

Check: Does this function have the point (2, 0) on it? Is this function not differentiable at x = 0?

$$\lim_{x \rightarrow \infty} \left(1 - e^{\frac{1}{2} - \frac{1}{x}} \right) = 1 - \sqrt{e} \quad \text{or} \quad 1 - e^{\frac{1}{2}}$$

$$f(x) = \frac{\ln x}{x} \quad \text{and} \quad f'(x) = \frac{1 - \ln x}{x^2}$$

To find the point at $x = e^2$, find $f(e^2)$

$$f(e^2) = \frac{\ln e^2}{e^2} = \frac{2 \ln e}{e^2} = \frac{2}{e^2}$$

To find the slope of the tangent line at $x = e^2$, find $f'(e^2)$

$$f'(e^2) = \frac{1 - \ln e^2}{(e^2)^2} = \frac{-1}{e^4}$$

Hence, the equation of the tangent line to the curve at $x = e^2$ is

$$y - \frac{2}{e^2} = -\frac{1}{e^4}(x - e^2)$$

To find the critical point, let $f'(x) = 0$

$$0 = \frac{1 - \ln x}{x^2} \text{ when } x = e$$

On the interval $(0, e)$ $f'(x) > 0$

On the interval (e, ∞) $f'(x) < 0$

Since $f'(x)$ changes from positive to negative values to $x = e$, the function f has a relative maximum at $x = e$.

Remember: You need to justify your answer with Calculus and more importantly, you need WORDS.

To find any points of inflection, we will need to find $f''(x)$

Which means we need to use the QUOTIENT RULE

$$f''(x) = \frac{(x^2)(-\frac{1}{x}) - (1 - \ln x)(2x)}{(x^2)^2}$$

Remember that $x > 0$ [given]

We have no choice but to simplify.

$$f''(x) = \frac{-x - 2x + 2x \ln x}{x^4}$$

FACTOR OUT GCF

$$f''(x) = \frac{x(-3 + 2 \ln x)}{x^4}$$

CANCEL

$$f''(x) = \frac{-3 + 2 \ln x}{x^3}$$

Now let $f''(x) = 0$ or $0 = -3 + 2 \ln x$

$$\begin{aligned} 3 &= 2 \ln x \\ \frac{3}{2} &= \ln x \\ e^{\frac{3}{2}} &= x \end{aligned}$$

On the interval $\left(0, e^{\frac{3}{2}}\right)$, $f''(x) < 0$

On the interval $\left(e^{\frac{3}{2}}, \infty\right)$, $f''(x) > 0$

Hence, thus, therefore, f has a point of inflection at $x = e^{\frac{3}{2}}$ because $f''(x)$ changes from negative to positive values at $x = e^{\frac{3}{2}}$

$$\lim_{x \rightarrow 0^+} \frac{\ln x}{x} = -\infty \text{ OR DOES NOT EXIST [both answers were accepted]}$$