

Calculus
Chapter 3.1 – 3.5 Review Sheet Solutions

1. f is differentiable on the interval $(-4, 4)$ except at $x = -2$ (corner), $x = -1/2$ (vertical tangent line), $x = 1$ (corner), $x = 2$ (corner), $x = 3$ (discontinuity)

2. To find marginal profit, find the derivative of $P \dots P'(x) = 12x^2 - 7$

The marginal profit of the 12th item (currently producing 11 items) is $P'(11) = 12(11)^2 - 7 = 1445$

3. A derivative at 2 exists when the derivative from the left and right sides are the same. Also, keep in mind that if the function is NOT continuous at 2, $f'(2)$ will not exist regardless of the left and right derivatives. (see notecards for definition of a derivative at $x = c$ and at an endpoint)

4. First find the derivative of the function: $f'(x) = \begin{cases} 4ax & x > 1 \\ -3 & x < 1 \end{cases}$. In order for the derivative to exist at $x = 1$, the

derivative from the left and the derivative from the right must be equal. So, $4a(1) = -3$, meaning $a = \frac{-3}{4}$.

In order for f to be continuous at $x = 1$, $\lim_{x \rightarrow 1} f(x) = f(1)$. In order for this limit to exist, $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x)$.

So, evaluating the left and right hand limits and substituting what we now know a equals we get

$$\begin{aligned} \lim_{x \rightarrow 1^-} f(x) &= \lim_{x \rightarrow 1^+} f(x) \\ -3(1) + 4 &= 2a(1)^2 + b \\ 1 &= 2\left(\frac{-3}{4}\right) + b \\ \frac{5}{2} &= b \end{aligned}$$

Therefore, to be continuous and differentiable, $a = \frac{-3}{4}$ and $b = \frac{5}{2}$.

5. a) Average velocity is $\frac{\text{total distance}}{\text{total time}} = \frac{x(3) - x(0)}{3 - 0} = \frac{-3 - 12}{3} = -5$ ft/s

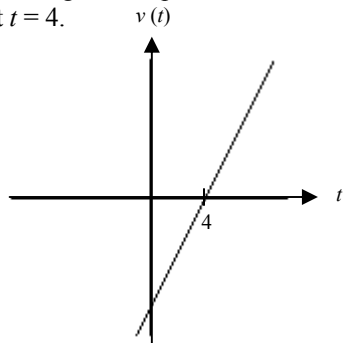
b) Velocity is the derivative of position: $v(t) = 2t - 8 \dots v(4) = 0$ ft/s

c) Since velocity is 0 at $t = 4$, the object is stopped at $t = 4$.

d) Acceleration is the derivative of velocity: $a(t) = 2 \frac{ft}{s^2}$.

e) The object changes direction when the velocity changes sign. There are two ways to look at this

Graphically: Notice at $t = 4$, the graph of the velocity function changes from negative to positive, so the object changes direction at $t = 4$.



Numerically: Since we already know the velocity is 0 at $t = 4$, then all we need to do is check the sign of the velocity function for values of t less than 4 and again for values of t greater than 4. Since $v(3) = -2$ and $v(5) = 2$, the object changes direction at $t = 4$.

f) The object slows down when the velocity and acceleration have opposite signs. Since the acceleration is always positive, then the object is slowing down when velocity is negative. Thus, when $t < 4$.

g) The object is moving left when velocity is negative. Again, when $t < 4$.

6. The equation of a tangent line requires a point and a slope. To get the point, plug $x = \frac{\pi}{2}$ into the function. So, the point is $(\frac{\pi}{2}, 0)$. To find the slope, take a derivative. Using the PRODUCT RULE (let $u = 2\sin x$ and $v = \cos x$) we have $y' = 2\sin x(-\sin x) + \cos x(2\cos x) = -2\sin^2 x + 2\cos^2 x$. When $x = \frac{\pi}{2}$, $y' = -2 - 0 = -2$. So, the equation of the tangent line is $y - 0 = -2(x - \frac{\pi}{2})$ or $y = -2x + \pi$ Try graphing both the function and the tangent line on your calculator as a way to check.

7. The first limit is the limit definition of the derivative of $\cos x$ at $x = \frac{\pi}{2}$. Since the derivative of $\cos x$ is $-\sin x$, then the answer is $-\sin(\frac{\pi}{2}) = -1$.

The second limit is the limit definition of the derivative of \sqrt{x} at $x = 4$. Since the derivative of \sqrt{x} is $\frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$, then the answer is $\frac{1}{2\sqrt{4}} = \frac{1}{4}$.

8. Using the alternative definition we get

$$\begin{aligned} f'(2) &= \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{\left(\frac{1}{x} - \frac{1}{2}\right)}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{\left(\frac{2-x}{2x}\right)}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{2-x}{2x} \cdot \frac{1}{x-2} \\ &= \lim_{x \rightarrow 2} \frac{-1}{2x} \\ &= \frac{-1}{4} \end{aligned}$$

You should be able to check this using the power rule ... since $f(x) = \frac{1}{x} = x^{-1}$, then

$$f'(x) = -1x^{-2} = \frac{-1}{x^2}. \text{ Therefore, } f'(2) = \frac{-1}{4}.$$

9. Using the alternative definition we get

$$\begin{aligned} f'(1) &= \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} \\ &= \lim_{x \rightarrow 1} \frac{(3x^2 + 5x) - (8)}{x - 1} \\ &= \lim_{x \rightarrow 1} \frac{3x^2 + 5x - 8}{x - 1} \\ &= \lim_{x \rightarrow 1} \frac{(3x + 8)(x - 1)}{x - 1} \\ &= \lim_{x \rightarrow 1} (3x + 8) \\ &= 11 \end{aligned}$$

You should be able to check this using the power rule ... since $f(x) = 3x^2 + 5x$, then

$$f'(x) = 6x + 5. \text{ Therefore, } f'(1) = 6(1) + 5 = 11.$$

10. $f'(3)$ is the derivative at $x=3$... or the rate of change in \$/min at 3 minutes. To approximate this value, use the slope of a secant line (any secant line that includes $x=3$) ... ANY of the following responses would be considered acceptable, as long as you explain what you are doing ...

$$\frac{f(4)-f(3)}{4-3} = \frac{11-9}{1} = 2$$

$$\frac{f(3)-f(2)}{3-2} = \frac{9-6}{1} = 3$$

$$\frac{f(4)-f(2)}{4-2} = \frac{11-6}{2} = \frac{5}{2}$$

11. The first worksheet left off the domain ... you should limit the domain of t to be $0 \leq t \leq 2\pi$ for this entire problem!

a) Velocity is the derivative of position (use product rule) ... $v(x) = s'(x) = x^2(\cos x) + \sin x(2x)$

b) The acceleration is the derivative of the velocity function (or second derivative of position) ... use the product rule for each part ... $a(x) = s''(x)$

$$= x^2(-\sin x) + \cos x(2x) + \sin x(2) + 2x(\cos x)$$

$$= -x^2 \sin x + 4x \cos x + 2 \sin x$$

c) The object is stopped when velocity equal 0 ... using a calculator the object is stopped at $x=0$, $x \approx 2.289$, and $x \approx 5.087$.

d) The particle changes direction when the velocity changes sign ... once again, using the graph, the velocity changes sign when $x \approx 2.289$ and $x \approx 5.087$.

e) The particle is speeding up when the velocity and acceleration have the same sign ... using your calculator, graph the acceleration function and find the zeros so that you can determine when the acceleration changes sign ... The zeros of the acceleration function are $x=0$, $x \approx 1.520$, and $x \approx 3.994$. Since velocity and acceleration are both positive on $(0, 1.520)$ and $(5.087, 2\pi)$ and they are both negative on $(2.289, 3.994)$ these are the three intervals for which the particle is speeding up

... Another way to do this is to consider that speed is the absolute value of velocity ... so graph the speed function and find the intervals where the speed is increasing (going up). ... to do this you would need to find the zeros and the maximums, but you would end up with the same intervals.

f) Zeros of $s(x)$ would be the same as the zeros of $\sin x$... $x=0, \pi, 2\pi$

g) Zeros of $v(x)$ are $x=0$, $x \approx 2.289$, and $x \approx 5.087$.

h) Zeros of $a(x)$ are $x=0$, $x \approx 1.520$, and $x \approx 3.994$.

12. If $f(x)$ has a derivative at $x=2$, then it must also be continuous at $x=2$... so a MUST BE TRUE (part of the definition of continuity), b MUST BE TRUE (because the $f'(2)$ is notation for the derivative at $x=2$), c is not necessarily true because we have been given nothing about the second derivative, d MUST BE TRUE (because if a function has a derivative at a point, then it must be continuous at that point), e MUST BE TRUE (because that is the alternative definition of the derivative at $x=2$), and f MUST BE TRUE (because that is the standard definition of the derivative at $x=2$).