Three forms of the limit definition of derivative

$$f'(c) = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x}$$

This gives us the $m_{\tan}$ at some value of $x = c, c \in$ Reals

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

$$\lim_{x \to 1} \frac{x^{100} - 1^{100}}{x - 1} = \frac{d}{dx} x^{100} \Big|_{x=1} = 100x^{99} \Big|_{x=1} = 100$$

This also gives us the $m_{\tan}$ at some value of $x = c, c \in$ Reals

SONG

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

This gives us a function rather than a numeric value.

You may use any form and you will need to be able to distinguish between numeric and algebraic answers.

Some examples to consider:

$$\lim_{x\to 2} \frac{x^5 - 32}{x-2}$$

$$= \frac{d}{dx}(x^5)\Big|_{x=2}$$

$$= 5x^4\Big|_{x=2} = 80$$

$f(x) = x^5$

$f'(2)$

$$\lim_{h\to 0} \frac{(x+h)^5 - x^5}{h}$$

$$= \frac{d}{dx}(x^5)$$

$$= 5x^4$$

$f(x) = x^5$

$f'(x)$

$\downarrow 2^5$

$$\lim_{h\to 0} \frac{(2+h)^5 - 32}{h}$$

$$= \frac{d}{dx} x^5\Big|_{x=2}$$

$$= 5x^4\Big|_{x=2} = 80$$

THINK $\lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$

$f(x) = x^5$

$f'(2)$ $c = 2$

$$\lim_{h\to 0}\frac{(x+h)^{11}-x^{11}}{h}$$

$$= \frac{d}{dx}x^{11}$$

$$= 11x^{10}$$

$f(x) = x^{11}$

$f'(x)$

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c} = f'(c)$$

$$\lim_{x\to 1}\frac{x^{11}-1}{x-1}$$

$$= \frac{d}{dx}x^{11}\Big|_{x=1}$$

$$= 11x^{10}\Big|_{x=1} = 11$$

$f(x) = x^{11}$

$f'(1)$

$$\lim_{h\to 0}\frac{(1+h)^{11}-1}{h}$$

$$= \frac{d}{dx}x^{11}\Big|_{x=1}$$

$$= 11x^{10}\Big|_{x=1}$$

$$= 11$$

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$

$f(x) = x^{11}$

$f'(1)$

$$\lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$$

$$= \frac{d}{dx}\cos x$$

$$= -\sin x$$

$f(x) = \cos x$

$f'(x)$

$$\lim_{x\to\frac{\pi}{2}}\frac{\cos x-\cos\left(\frac{\pi}{2}\right)}{x-\frac{\pi}{2}}$$

$$= \frac{d}{dx}\cos x\Big|_{x=\frac{\pi}{2}}$$

$$= -\sin x\Big|_{x=\frac{\pi}{2}} = -1$$

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

$f(x)=\cos x$

$f'\left(\frac{\pi}{2}\right)$

$$\lim_{h\to 0}\frac{\cos\left(\frac{\pi}{2}+h\right)-\cos\left(\frac{\pi}{2}\right)}{h}$$

$$= \frac{d}{dx}\cos x\Big|_{x=\frac{\pi}{2}}$$

$$= -\sin x\Big|_{x=\frac{\pi}{2}} = -1$$

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$

$f(x)=\cos x$

$f'\left(\frac{\pi}{2}\right)$

Another use for these forms has to do with finding out if a function is differentiable at a point.

To say that a function is differentiable, means that the derivative exists. Just because a function is continuous does not mean that it is differentiable at all points on its domain. For example, $f(x)=\sqrt{x}$ is continuous on $[0,\infty]$ but not differentiable at $x=0$.

One way to tell if a function is differentiable at a point is to use the second form of $f'(c)$.

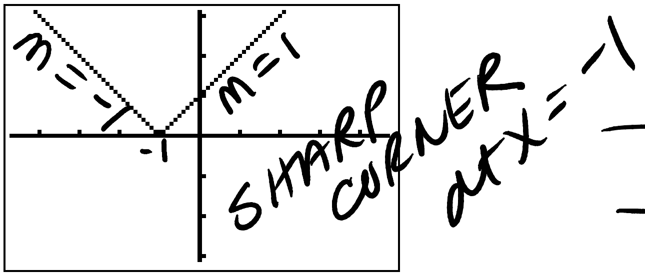
$$\lim_{x \to c^-} f'(x) = \lim_{x \to c^+} f'(x)$$

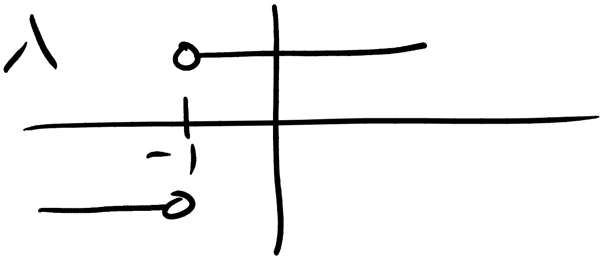
$$\lim_{x \to c^-} \frac{f(x)-f(c)}{x-c} = \lim_{x \to c^+} \frac{f(x)-f(c)}{x-c}$$

These must exist AND be equal in order for $f'(c)$ to exist.

Consider $g(x)=|x+1|$

Domain:





Graph of $g(x)$ [Let's first draw graph of the derivative]
Does $g'(-1)$ exist?
Let's look at the right- and left-hand limits.

$$\lim_{x\to -1^-} \frac{g(x)-g(-1)}{x-(-1)}$$

Left-hand limit $\lim_{x\to -1^-} g'(x) = -1$

$$= \lim_{x\to -1^-} \frac{|x+1|-0}{x+1} = -1$$

[It helps to let x be some number very close to -1 on the left, such as x= -1.001]

$$\lim_{x\to -1^+} \frac{f(x)-f(-1)}{x-(-1)}$$

Right-hand limit $\lim_{x\to -1^+} g'(x) = 1$

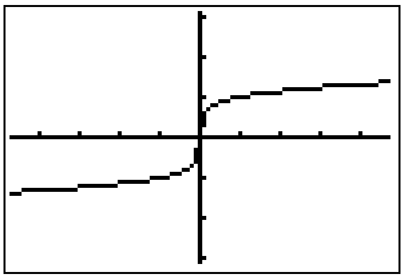
$$= \lim_{x\to -1^+} \frac{|x+1|-0}{x+1} = 1$$

[It helps to let x be some number very close to -1 on the right, such as x= -0.99]

Since the right- and left-hand limits do not agree at x= -1, then $g'(-1)$ does not exist.

Does the derivative exist at $x = 2$?
At $x = 2$ we can find the instantaneous rate of change by finding the slope of the line. In this case, $f'(2) = 1$

Now consider $h(x) = x^{\frac{1}{5}}$ at $x = 0$



Domain of $h(x)$: $(-\infty, \infty)$

$h(x) = x^{\frac{1}{5}}$

$h'(x) = \frac{1}{5}x^{-4/5}$

$= \frac{1}{5x^{4/5}}$

Does $h(0)$ exist? Yes, $h(0) = 0$
How about $h'(0)$?

$\lim_{x \to 0^-} h'(x) = \infty$

$\lim_{x \to 0^+} h'(x) = \infty$

$h'(0)$ dne

$$\lim_{x \to 0^-} \frac{h(x) - h(0)}{x - 0} = \infty$$
$$\lim_{x \to 0^+} \frac{h(x) - h(0)}{x - 0} = \infty$$

Hence, we have a vertical tangent at $x = 0$, so $h(x)$ is NOT differentiable at $x = 0$

How about $f(x) = \frac{1}{x}$ at $x = 0$?

Does $f(0)$ exist? Nope! So how can we have a tangent line at an undefined point? You can't. $f'(0)$ dne because $f(x)$ is NOT continuous at $x=0$

See page 106 #81 through 86.

(81) $f'(-1)$ dne ; $f(1)$ undefined [NOT continuous]

(82) $f'(3)$, $f'(-3)$ dne ; f has sharp corners OUCH!

(83) $f'(3)$ dne ; f has a cusp (OUCH!)

(84) $f'(2)$ and $f'(-2)$ dne
f is NOT continuous at $x = \pm 2$

(85) $f'(1)$ dne because vertical tangent at $x=1$

(86) $f'(0)$ dne because $f(x)$ has a jump discontinuous at $x=0$

So, based on what we've seen today, non-differentiable points will occur if we encounter:

(1) discontinuity – no point, no tangent line
(2) sharp corner or cusp – right- and left-hand limits of the derivative will not agree
(3) vertical tangents – slope of a vertical line is undefined

Homework project – do first page [yes, both sides] of the handout called:
2009weirdchapter2.doc