Homogeneous Diff EQ

These are not the same as the separable differential equations from AP Calculus BC . Bummer! No worries. They are not that difficult.

$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ which can be easily solved by using the substitution $y = vx$ where v is a function of x. This is called the change of variables method.

This substitution will rewrite the differential equation to a separable form. Here is an example.

Use the substitution $y = vx$, where v is a function of x to solve:

Find a solution to the differential equation $\frac{dy}{dx} = \frac{y}{x} - 2$ that passes through the point (1,1)

Ack! We can't separate the way we did in AP Calculus! We need to learn this method.

$y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

[Implicit differentiation with the product rule]

And, $\frac{y}{x} = v$

Now let's go back to the original equation

$\frac{dy}{dx} = \frac{y}{x} - 2$ Start substituting

$$v + x\frac{dv}{dx} = v - 2$$

$$x\frac{dv}{dx} = -2$$

$$\frac{dv}{dx} = \frac{-2}{x}$$

Now we can do what you learned in AP Calculus

Separate

$$\int dv = \int \frac{-2}{x}\ dx$$

Integrate

$$v \quad = \quad -2\ln x + C$$

$$\frac{y}{x} \quad = \quad -2\ln x + C$$

Solve for C given that the point (1, 1) in on the curve

$$\frac{1}{1} \quad = -2\ln 1 + C$$ Hence C=1

Solve for y

$$y = x(1 - 2\ln x)$$

Example 2

Find a solution to the differential equation $\frac{dy}{dx} = 3 - \frac{y}{x}$ that passes through the point (3,12)

Example 3

Find the particular solution for $xy^2 \dfrac{dy}{dx} = x^3 + y^3$ given $y = 3$ when $x = 1$.

From a previous Paper 3 Examination

Solve the differential equation

$y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$x^2 \frac{dy}{dx} = y^2 + xy + 4x^2,$$

given that $y = 2$ when $x = 1$. Give your answer in the form $y = f(x)$.

$$\frac{dy}{dx} = \frac{y^2}{x^2} + \frac{xy}{x^2} + \frac{4x^2}{x^2}$$

$$v + x\frac{dv}{dx} = v^2 + v + 4$$

$$\int \frac{dv}{v^2+4} = \int \frac{1}{x}\,dx$$

$$\frac{1}{2}\arctan\left(\frac{v}{2}\right) = \ln|x| + C$$

$$\frac{1}{2}\arctan\left(\frac{y}{2x}\right) = \ln|x| + C$$

Point $(1,2)$

$$\frac{1}{2}\arctan\left(\frac{2}{2}\right) = \ln|1| + C$$

Hence $C = \frac{\pi}{8}$

$$\frac{1}{2}\arctan\left(\frac{y}{2x}\right) = \ln|x| + \frac{\pi}{8}$$

$$\arctan\left(\frac{y}{2x}\right) = 2\ln|x| + \frac{\pi}{4}$$

$$\frac{y}{2x} = \tan\left(2\ln x + \frac{\pi}{4}\right)$$

$$y = 2x\tan\left(2\ln x + \frac{\pi}{4}\right)$$

Now find the general solution for:

$\frac{dy}{dx} = \frac{x-y}{x}$ using the substitution $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\frac{y}{x} = v$$

$$\frac{dy}{dx} = \frac{x}{x} - \frac{y}{x}$$

$$v + x\frac{dv}{dx} = 1 - v$$

$$x\frac{dv}{dx} = 1 - 2v$$

$$\int \frac{dv}{1-2v} = \int \frac{1}{x}\,dx$$

$$-\frac{1}{2}\ln|1-2v| = \ln|x| + C$$

$$\ln|1-2v| = -2(\ln|x| + C)$$

$$e^{\ln|1-2v|} = e^{-2(\ln|x|+C)}$$

$$1 - 2v = Ce^{-2\ln x}$$

$$1 - \frac{2y}{x} = Ce^{-2\ln x}$$

$$y = -\frac{1}{2}\left(\frac{C}{x} - x\right)$$