Homogeneous Diff EQ

These are not the same as the separable differential equations from AP Calculus BC . Bummer! No worries. They are not that difficult.

$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ which can be easily solved by using the substitution $y = vx$ where v is a function of x. This is called the change of variables method.

This substitution will rewrite the differential equation to a separable form. Here is an example.

Use the substitution $y = vx$, where v is a function of x to solve:

Find a solution to the differential equation $\frac{dy}{dx} = \frac{y}{x} - 2$ that passes through the point (1,1)

Ack! We can't separate the way we did in AP Calculus! We need to learn this method.

$y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

[Implicit differentiation with the product rule]

And, $\frac{y}{x} = v$

Now let's go back to the original equation

$\frac{dy}{dx} = \frac{y}{x} - 2$ Start substituting

$$\cancel{v} + x\frac{dv}{dx} = \cancel{v} - 2$$

$$x\frac{dv}{dx} = -2$$

$$\frac{dv}{dx} = \frac{-2}{x}$$

Now we can do what you learned in AP Calculus

Separate

$$\int dv = \int \frac{-2}{x}\ dx$$

Integrate

$$v \quad = \quad -2\ln x + C$$

$$\frac{y}{x} \quad = \quad -2\ln x + C$$

Solve for C given that the point (1, 1) in on the curve

$$\frac{1}{1} \quad = -2\ln 1 + C$$ Hence C=1

Solve for y

$$y = x(1-2\ln x)$$

Example 2

Find a solution to the differential equation $\frac{dy}{dx} = 3 - \frac{y}{x}$ that passes through the point (3,12)

$y = vx$ Hence $\frac{y}{x} = v$

AND $\frac{d}{dx} y = \frac{d}{dx} vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

my substitution stuff ♡ CARDIOD

$12 = v(3)$
$4 = v$

$v + x\frac{dv}{dx} = 3 - v$

$x\frac{dv}{dx} = 3 - 2v$

$\int \frac{1}{3-2v}\,dv = \int \frac{1}{x}\,dx$

$-\frac{1}{2}\ln|3-2v| = \ln|x| + C_1$

$\ln|3-2v| = -2\ln|x| + C_2$

use (x, v) $(3,4)$

$\ln|-5| = -2\ln 3 + C_2$

$\ln 5 + 2\ln 3 = C_2$

$\ln 45 = C_2$

$\ln|3-2v| = -2\ln|x| + \ln 45$

$e^{\ln|3-2v|} = e^{-2\ln|x| + \ln 45}$

$$2v - 3 = \frac{45}{x^2}$$

$$2v = \frac{45}{x^2} + 3$$

$$2\left(\frac{y}{x}\right) = \frac{45}{x^2} + 3$$

Example 3

$$y = \frac{1}{2}\left(\frac{45}{x} + 3x\right)$$

Find the particular solution for $xy^2 \frac{dy}{dx} = x^3 + y^3$ given $y = 3$ when $x = 1$.

let $y = vx$ [SAME AS BEFORE]

$$\frac{dy}{dx} = \frac{x^3 + y^3}{xy^2}$$

$$\frac{dy}{dx} = \frac{x^2}{y^2} + \frac{y}{x}$$

NOW SUBSTITUTE "V" STUFF

$$v + x\frac{dv}{dx} = \frac{1}{v^2} + v$$

$$x\frac{dv}{dx} = \frac{1}{v^2}$$

$$\int v^2 dv = \int \frac{1}{x} dx$$

$$\frac{v^3}{3} = \ln|x| + C$$

$$\frac{1}{3}\left(\frac{y^3}{x^3}\right) = \ln|x| + C$$

$$\frac{1}{3}\left(\frac{3^3}{1^3}\right) = 0 + C$$

$C = 9$

$$\frac{1}{3}\left(\frac{y^3}{x^3}\right) = \ln|x| + 9$$

$$\frac{y^3}{x^3} = 3\ln|x| + 27$$

$$y^3 = 3x^3\ln|x| + 27x^3$$

$y = \sqrt[3]{3x^3 \ln|x| + 27x^3}$

$y = x\left(\sqrt[3]{3 \ln|x| + 27}\right.$

From a previous Paper 3 examination

Solve the differential equation

$$x^2 \frac{dy}{dx} = y^2 + xy + 4x^2,$$

given that $y = 2$ when $x = 1$. Give your answer in the form $y = f(x)$.

Now find the general solution for:

$\frac{dy}{dx} = \frac{x - y}{x}$ using the substitution $y = vx$