Moving on with vectors:

Determining the angle between two given vectors

The most common formula used is found in your handy-dandy formula packet. It is:

$$\cos\theta = \frac{v_1 w_1 + v_2 w_2 + v_3 w_3}{|v||w|} \text{ where } v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \text{ and } w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

Or, $\cos\theta = \dfrac{v \bullet w}{|v||w|}$

Of course, if the vectors are perpendicular to each other, then dot product is equal to zero.

Also given in your formula packet is the formula:

$\dfrac{|v \times w|}{|v||w|} = \sin\theta$ where θ is the angle between v and w

Let's just try a few!

Find the angle between the vectors $u = i - 2j + 2k$ and $v = -3i + 6j + 2k$

$$\cos\theta = \frac{u \cdot v}{|u||v|}$$

$$= \frac{-3 - 12 + 4}{\sqrt{9}\sqrt{49}}$$

$$\cos\theta = -\frac{11}{21}$$

$$\theta \approx 2.12 \text{ RADIANS} \quad \text{or} \approx 122^\circ$$

Determine which, if any, of the following vectors are orthogonal. [That means "perpendicular"]

$u = 7i + 3j + 2k$

$v = -3i + 5j + 3k$

$w = i + k$

JUST FIND the DOT PRODUCTS

$u \cdot v = 7(-3) + 3(5) + 2(3) = 0$ ☆

$u \cdot w = 7(1) + 3(0) + 2(1) = 9$

$v \cdot w = -3(1) + 5(0) + 3(1) = 0$ ☆

Hence v and u are orthogonal AND v and w are orthogonal

$A(1, 2, 3)$, $B(-3, 2, 4)$, $C(1, -4, 3)$ are the vertices of a triangle. Show that the triangle is a right triangle and find its area.

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} \quad \text{or} \quad \overrightarrow{OB} - \overrightarrow{OA}$$
$$= -4i + k$$

$$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$$
$$= -6j$$

$$\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC}$$
$$= 4i - 6j - k$$

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -6 \\ 0 \end{pmatrix}$$
$$= 0 + 0 + 0$$
$$= 0$$

Hence: ΔABC is A RIGHT Δ

$$\text{area of } \Delta ABC = \frac{1}{2} |\overrightarrow{AB}| |\overrightarrow{AC}|$$
$$= \frac{1}{2} \sqrt{(-4)^2 + 1^2} \sqrt{(-6)^2}$$
$$= 3\sqrt{17}$$

At the end of this document is the page of vector "stuff" from the formula packet

Show that line $\dfrac{x-2}{1}=\dfrac{y-2}{3}=\dfrac{z-3}{1}$ and $\dfrac{x-2}{1}=\dfrac{y-3}{4}=\dfrac{z-4}{2}$ intersect and find the coordinate of P, the point of intersection.

L_1: $\dfrac{x-2}{1}=\dfrac{y-2}{3}=\dfrac{z-3}{1}=\lambda$

$x=2+\lambda$

$y=2+3\lambda$

$z=3+\lambda$

L_2: $\dfrac{x-2}{1}=\dfrac{y-3}{4}=\dfrac{z-4}{2}=\mu$

$x=2+\mu$

$y=3+4\mu$

$z=4+2\mu$

$2+\lambda=2+\mu$

$2+3\lambda=3+4\mu$

$3+\lambda=4+2\mu$

$\lambda=\mu$

$2+3\mu=3+4\mu$

$\lambda=-1=\mu$

$P=\begin{pmatrix}2\\3\\4\end{pmatrix}+-1\begin{pmatrix}1\\4\\2\end{pmatrix}$

$P=(1,-1,2)$

(b) Find the Cartesian equation of the plane Π that contains the two lines

$$\begin{pmatrix} i & j & k \\ 1 & 3 & 1 \\ 1 & 4 & 2 \end{pmatrix}$$

$$= i\,(3(2) - 4(1)) - j\,(1(2) - 1(1))$$
$$+ k\,(1(4) - 3(1))$$
$$= 2i - j + k$$

$$r = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Cartesian

$$2x - y + z = c$$
$$2(1) - (-1) + 2 = c$$
$$2x - y + z = 5$$

(c) The point $Q(3, 4, 3)$ lies on Π. The line L passes through the midpoint of $[PQ]$. Point S is on L such that $\left|\overrightarrow{PS}\right| = \left|\overrightarrow{QS}\right| = 3$, and the triangle PQS is normal to the plane Π. Given that there are two possible positions for S, find their coordinates.

M is midpoint of $PQ = \left(\frac{1+3}{2}, \frac{-1+4}{2}, \frac{2+3}{2}\right)$

$M = \left(2, \frac{3}{2}, \frac{5}{2}\right)$

DIRECTION OF $\overrightarrow{MS}$ is SAME as Π

thus $2i - j + k$

general POINT R on MS

$\left(2+2\lambda, \frac{3}{2} - \lambda, \frac{5}{2} + \lambda\right)$

$\overrightarrow{PR} = \overrightarrow{PO} + \overrightarrow{OR}$

$= \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 2+2\lambda \\ \frac{3}{2} - \lambda \\ \frac{5}{2} + \lambda \end{pmatrix}$

$PR = (1+2\lambda)i + \left(\frac{5}{2} - \lambda\right)j + \left(\frac{1}{2}+\lambda\right)k$

5.1	Magnitude of a vector	$\lvert v \rvert = \sqrt{v_1^{\,2} + v_2^{\,2} + v_3^{\,2}}$, where $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$
	Distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2)	$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$
	Coordinates of the midpoint of a line segment with endpoints (x_1, y_1, z_1), (x_2, y_2, z_2)	$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$
5.2	Scalar product	$\boldsymbol{v} \cdot \boldsymbol{w} = \lvert \boldsymbol{v} \rvert \lvert \boldsymbol{w} \rvert \cos\theta$, where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ $\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$, where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$
	Angle between two vectors	$\cos\theta = \frac{v_1 w_1 + v_2 w_2 + v_3 w_3}{\lvert \boldsymbol{v} \rvert \lvert \boldsymbol{w} \rvert}$
5.3	Vector equation of a line	$\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$
	Parametric form of equations of a line	$x = x_0 + \lambda l,\ y = y_0 + \lambda m,\ z = z_0 + \lambda n$
	Cartesian equations of a line	$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$
5.5	Vector product (Determinant representation)	$\boldsymbol{v} \times \boldsymbol{w} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$ $\lvert \boldsymbol{v} \times \boldsymbol{w} \rvert = \lvert \boldsymbol{v} \rvert \lvert \boldsymbol{w} \rvert \sin\theta$, where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$
	Area of a triangle	$A = \frac{1}{2} \lvert \boldsymbol{v} \times \boldsymbol{w} \rvert$
5.6	Vector equation of a plane	$\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b} + \mu \boldsymbol{c}$
	Equation of a plane (using the normal vector)	$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$
	Cartesian equation of a plane	$ax + by + cz + d = 0$