Series From: http://www2.guhsd.net/algebra2/

When the terms of a sequence are added, the resulting expression is a **series**. A series can be finite or infinite.

FINITE SEQUENCE	INFINITE SEQUENCE
3, 6, 9, 12, 15	3, 6, 9, 12, 15, . . .
FINITE SERIES	**INFINITE SERIES**
$3 + 6 + 9 + 12 + 15$	$3 + 6 + 9 + 12 + 15 + \ldots$

The sum of the first n terms of a sequence is:

$S_n = u_1 + u_2 + u_3 + \ldots + u_n$

For example:

Find the sum of the first five terms of the series $\{3n\}$ for $n = 1, 2, 3, 4, 5$

Our arithmetic sequence is: 3, 6, 9, 12, 15

Our arithmetic series is:

$S_5 = 3 + 6 + 9 + 12 + 15$

$S_5 = 45$

An arithmetic series is the addition of successive terms of an arithmetic sequence.

The sum of an arithmetic series can be found with this convenient formula [which is in our formula packet]

$$S_n = \frac{n}{2}(u_1 + u_n)$$

Let's see if it works:

$$\sum_{1}^{100} n =$$

$S_n = 1 + 2 + 3 + \ldots + 98 + 99 + 100$

In this problem, $n = 100$, $u_1 = 1$, $u_{100} = 100$

$$S_{100} = \frac{100}{2}(1 + 100)$$
$$= 5050$$

Note: If you don't know the last term of the sequence, then you can always use our handy formula $u_n = u_1 + (n-1)d$

So, an alternate formula for an arithmetic series would be:

$S_n = \frac{n}{2}(u_1 + u_n)$ now substitute $u_n = u_1 + (n-1)d$

$S_n = \frac{n}{2}(2u_1 + (n-1)d)$

$$S_n = \frac{n}{2}(u_1 + u_1 + (n-1)d)$$
$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$

Let's try some problems!

(1) Find the number of terms in the following series and then find the sum

1+3+5+7+…+99

This is an arithmetic series with $u_1 = 1$ and $u_n = 99$

The value of $d = 2$

Let's find n first:

$$u_n = u_1 + (n-1)d$$
$$99 = 1 + 2(n-1)$$
$$98 = 2n - 2$$
$$50 = n$$

Now we can find the sum.

We can use $$S_n = \frac{n}{2}(u_1 + u_n)$$

$$S_{50} = \frac{50}{2}(1+99)$$
$$S_{50} = 2500$$

(2) Find the sum of 8+11+14+17+… to 50 terms

This is an arithmetic series with $u_1 = 8$ and $d = 3$
We know that $n = 50$, so we can either find u_{50} or just use our alternate formula.

Let's try our alternate formula!

$$S_n = \frac{n}{2}\left(2u_1 + (n-1)d\right)$$

$$S_{50} = \frac{50}{2}\left(2(8) + (50-1)(3)\right)$$

$$= 4075$$

(3) An arithmetic sequence has a common difference of 7 and $u_{11} = 53$. Find S_{19}.

We'll need to find u_1 first because it is necessary in either formula.
We can use: $u_n = u_1 + (n-1)d$

$$u_{11} = 53 = u_1 + (11-1)(7)$$

$$-17 = u_1$$

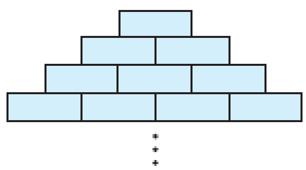
Now we can use $S_n = \frac{n}{2}\left(2u_1 + (n-1)d\right)$ to find S_{19}

$$S_{19} = \frac{19}{2}\left(2(-17) + (19-1)(7)\right)$$

$$= 874$$

Let's take a look at the following problem:

5



A bricklayer builds a triangular wall with layers of bricks as shown. If the bricklayer uses 171 bricks, how many layers are placed?

We know that $S_n = 171$, $d = 1$, and $u_1 = 1$. We need to find n.

$$S_n = \frac{n}{2}\left(2u_1 + (n-1)d\right)$$

$$171 = \frac{n}{2}\left(2(1) + (n-1)(1)\right)$$

$$171 = \frac{n}{2}(n+1)$$

$$342 = n^2 + n$$

$$0 = n^2 + n - 342$$

$$0 = (n-18)(n+19)$$

$$n = 18$$

And now for something slightly different, let's consider the following:

Prove that the sum of the first n positive integers is $\frac{n(n+1)}{2}$ or show that $1+2+3+...+n=\frac{n(n+1)}{2}$

[But not with PMI]

We know that $d=1$, $u_1=1$, and that there are n integers

$$S_n=\frac{n}{2}\left(2u_1+(n-1)d\right)$$

$$S_n=\frac{n}{2}\left(2(1)+(n-1)(1)\right)$$

$$=\frac{n}{2}(n+1)$$

$$=\frac{n(n+1)}{2}$$

FTW!

A geometric series is the addition of successive terms of a geometric sequence.

Can we use the formula from the arithmetic series? NO!

Let $\{u_n\}$ be a geometric sequence.

Then, the finite geometric series is:

$S_n = u_1 + u_2 + u_3 + \ldots + u_n$ which we can rewrite as

$S_n = u_1 + u_1 r + u_1 r^2 + u_1 r^3 + \ldots + u_1 r^{n-1}$

Through algebraic "magic", we can multiply both sides by r to get:

$r S_n = r u_1 + r^2 u_1 + r^3 u_1 + r^4 u_1 + \ldots + r^n u_1$

Now let's use some more algebraic "magic" and subtract $S_n - r S_n$ which will equal

$$\left(u_1 + r u_1 + r^2 u_1 + r^3 u_1 + \ldots + r^{n-1} u_1\right) - \left(r u_1 + r^2 u_1 + r^3 u_1 + \ldots + r^n u_1\right)$$

Cancel where you can!

Our result it:

$S_n - r S_n = u_1 - r^n u_1$ Now let's factor the left side

$S_n (1 - r) = u_1 - r^n u_1$ Now isolate S_n

$$S_n = \frac{u_1 - r^n u_1}{1 - r} \quad \text{or} \quad \frac{u_1\left(1 - r^n\right)}{1 - r} \quad \text{or} \quad \frac{a\left(1 - r^n\right)}{1 - r} \quad [r \neq 1]$$

If $r > 1$, then both the numerator and denominator are negative, and it might be more convenient to use this formula:

$$S_n = \frac{a\left(r^n - 1\right)}{r - 1}$$

Example 16

Find the sum of $2 + 6 + 18 + 54 + \ldots$ to 12 terms.

The series is geometric with $u_1 = 2$, $r = 3$ and $n = 12$.

So, $S_{12} = \dfrac{2(3^{12} - 1)}{3 - 1}$ $\quad\left\{\text{Using } S_n = \dfrac{u_1(r^n - 1)}{r - 1}\right\}$

$= \dfrac{2(3^{12} - 1)}{2}$

$= 531\,440$

The Sum of an Infinite Geometric Series

When $|r| < 1$, the terms of a geometric series decrease as n increases. In this case, a series with an infinite number of terms has a finite sum and we can say that as $n \to \infty$, the sum converges. This sum, S_∞, can be found with the following formula:

$$S_\infty = \frac{u_1}{1-r} \text{ or } \frac{a}{1-r} \qquad |r| < 1$$

Note: If $|r| > 1$, then the infinite series diverges and has no sum.

Consider: $S_n = \dfrac{1}{3} + \dfrac{1}{9} + \dfrac{1}{27} + \ldots + \dfrac{1}{3^n}$ Find S_n as $n \to \infty$

$$S_\infty = \frac{u_1}{1-r} \quad \text{where } u_1 = \frac{1}{3} \text{ and } r = \frac{1}{3}$$

CHAI!

$$S_\infty = \frac{\frac{1}{3}}{1-\frac{1}{3}}$$

$|r| < 1$

$$S_\infty = \frac{\frac{1}{3}}{\frac{2}{3}}$$

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$$S_\infty = \frac{1}{2}$$