First Order Linear Differential Equation and the Integrating Factor Method

So far you have conquered separable differential equations and the "change of variable" or "v-sub" method for homogeneous differential equations. Now for something slightly different.

A first order linear differential equation has the form $\frac{dy}{dx} + P(x)\,y = Q(x)$

Notice that the y term is to the first power [hence, linear] and that $\frac{dy}{dx}$ is also a part of the equation. P and Q are continuous functions.

To solve a linear differential equation, you first must put the equation in the form above to figure out what P and Q are, and then integrate $P(x)$ to get $u(x) = e^{\int P(x)dx}$. Our solution should be in the form of $ye^{\int P(x)dx} = \frac{1}{u(x)} \int Q(x)\,e^{\int P(x)dx}\,dx + C$

Or $\boxed{y = \frac{1}{u(x)}\int Q(x)u(x)dx}$

$u(x) = e^{\int P(x)dx}$

Here is an easy one:

$\frac{dy}{dx} + y = e^x$ Notice: y is to the first degree

$\frac{dy}{dx} + P(x)\,y = Q(x)$ Here is the standard form

So, $P(x) = 1$ and $Q(x) = e^x$ Awesome!

Our integrating factor is $u(x) = e^{\int P(x)dx}$

Hence, $u(x) = e^{\int 1 dx} = e^x$

Thus, our general solution is:

$u(x) = e^{\int P(x)dx}$

$$y = \frac{1}{u(x)}\int Q(x)u(x)\,dx$$

$$y = \frac{1}{e^x}\int e^x\left(e^x\right)dx$$

$\int (e^x)(e^x) = e^{2x}$

$$y = e^{-x}\left(\left(\tfrac{1}{2}e^{2x}\right) + C\right) = \tfrac{1}{2}e^x + Ce^{-x}$$

Example 2

Find the general solution of

$$x\frac{dy}{dx} - 2y = x^2$$

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

We'll need a little algebra to get this into the standard form

$$u(x) = e^{\int P(x)dx}$$

$$\frac{dy}{dx} + \left(\frac{-2}{x}\right)y = x$$

$$= e^{-2\int \frac{1}{x}dx}$$

$$y = \frac{1}{u(x)}\int Q(x)u(x)\,dx$$

$$= e^{-2\ln|x|}$$

$$y = x^2 \int \frac{1}{x}\,dx$$

$$= \frac{1}{x^2}$$

$$y = x^2\left[\ln|x| + C\right]$$

$$y = x^2\ln|x| + Cx^2$$

Example 3

$$\frac{dy}{dx} - y\tan x = 1 \text{ for } \tfrac{-\pi}{2} < x < \tfrac{\pi}{2}$$

$$u(x) = e^{\int P(x)dx}$$
$$= e^{\int -\tan x\, dx}$$
$$= e^{\ln|\cos x|}$$
$$= |\cos x|$$

$$y = \frac{1}{|\cos x|} \int |\cos x|\, dx$$
$$= \sec x(\sin x + C)$$
$$= \tan x + C\sec x$$

You try:

$$(y+1)\cos x\,dx - dy = 0$$

Remember that you must put into standard form first

$$\frac{dy}{dx} + -y\cos x = \cos x$$

$$y = \frac{1}{u(x)}\int Q(x)u(x)\,dx$$

$$u(x) = e^{\int -\cos x\,dx} = e^{-\sin x}$$

$$y = \frac{1}{e^{-\sin x}}\int \cos x\, e^{-\sin x}\,dx$$

$$y = \frac{1}{e^{-\sin x}}\left(-e^{-\sin x} + C\right)$$

$$y = -1 + \frac{C}{e^{-\sin x}}$$

$$\frac{dy}{dx} - 3x^2 y = e^{x^3}$$

$$u(x) = e^{\int -3x^2\,dx}$$

$$= e^{-x^3}$$

$$y = \frac{1}{u(x)} \int Q(x)u(x)\,dx$$

$$= \frac{1}{e^{-x^3}} \left(\int e^{x^3} e^{-x^3}\,dx \right)$$

$$= \frac{1}{e^{-x^3}} (x + C)$$

$$y = \frac{x}{e^{-x^3}} + \frac{C}{e^{-x^3}}$$

We will do some from past IB exams next week.