Continuous Probability Density Functions

'Kay so, we have spent a lot of time on discrete random variables. Now we need to work with continuous random variables.

Tons of stuff in real-life uses continuous values like weight, height, time, etc. For a continuous random variable *X, x* can take any real value. The function used to specify the probability distribution is aptly named the probability density function.

A continuous probability generating function $f(x)$ is a function where $f(x) \geq 0$ on a given interval, $[a,b]$ AND $\int_a^b f(x)\, dx = 1$

And now for the important definitions!

The mode is the value of x, at the maximum value of $f(x)$ on $[a,b]$.

The median m, is the solution for m of the equation

$$\int_a^m f(x)dx = \frac{1}{2}$$

The mean μ or $E(X)$ is defined at $\mu = \int_a^b x\, f(x)dx$

$$Var(X) = E\left(X^2\right) - \{E(X)\}^2 = \left[\int_a^b x^2 f(x)\,dx\right] - \mu^2$$

As you can see, we get to use a lot of Calculus! Yay!

Our textbook's first example on page 748:

$$f(x) = \begin{cases} \frac{1}{2}x & on\ [0,2] \\ 0 & elsewhere \end{cases}$$ is a p.d.f

Check to verify the statement then find the mode, median, mean, Var(X), and σ

To check that $f(x)$ really is a p.d.f. we need to make sure that $1 = \int_0^2 \frac{1}{2} x \, dx$

$$\int_0^2 \frac{1}{2} x \, dx$$

$$= \frac{1}{4} x^2 \Big|_0^2$$

$$= \frac{1}{4} [4 - 0]$$

Mode will be the maximum of $f(x)$

$$x = 2$$

To find the median, *m*, we need to find the value of *m* such that:

$$\int_0^m \frac{1}{2} x \, dx = \frac{1}{2}$$

$$\frac{1}{4} x^2 \Big|_0^m = \frac{1}{2}$$

$$m^2 - 0 = 2$$

$$m = \sqrt{2}$$

To find $\mu = \int_0^2 x\, f(x)\, dx$

So, $\mu = \int_0^2 \frac{1}{2} x^2 dx$

$$\mu = \frac{x^3}{6} \Big|_0^2$$

$$\mu = \frac{4}{3}$$

To find $Var(X)$ we need to find $\int_0^2 x^2 f(x)\, dx$

So, $Var(X) = \int_0^2 \frac{1}{2} x^3 dx - \{E(X)\}^2$

$$= \frac{x^4}{8} \Big|_0^2 - \left(\frac{4}{3}\right)^2$$

$$= \frac{2}{9}$$

And to find σ we just need $\sqrt{Var(X)}$ so, $\sigma = \frac{\sqrt{2}}{3}$

Let's just try a few on page 749;

1.

$$f(x) = \begin{cases} ax(x-4), & 0 \leqslant x \leqslant 4 \\ 0 & \text{elsewhere} \end{cases}$$ is a continuous probability density function.

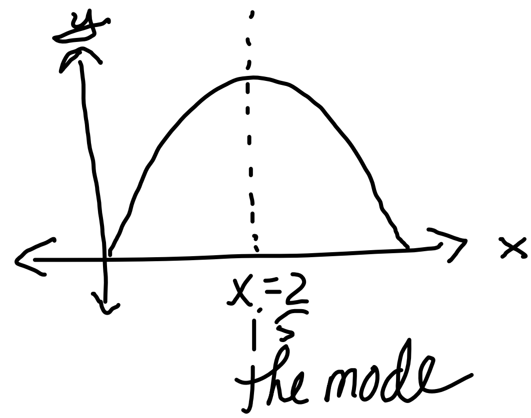
a Find a. **b** Sketch the graph of $y = f(x)$.

c Find: **i** the mean **ii** the mode **iii** the median **iv** the variance.

(a) $a\int_0^4 (x^2-4x)\,dx = 1$

$a\left[\frac{x^3}{3}-2x^2\right]_0^4 = 1$

$a = -\frac{3}{32}$



$\mu = -\frac{3}{32}\int_0^4 (x^3-4x^2)\,dx$

$\mu = 2$

$\int_0^m -\frac{3}{32}(x^2-4x)\,dx = \frac{1}{2}$

$\frac{m^3}{3} - 2m^2 = -\frac{16}{3}$

$m = 2$ since $0 < m < 4$

$VAR(x) = \left[\int_0^4 x^2 f(x)\,dx\right] - \mu^2$

$= 4.8 - 2^2$

$= .8$

$\sigma = \sqrt{.8}$

#3.

$$f(x) = \begin{cases} ke^{-x}, & 0 \leqslant x \leqslant 3 \\ 0 & \text{elsewhere} \end{cases}$$ is a probability density function.

a Find k to 4 decimal places. **b** Find the median.

$$k\int_0^3 e^{-x}\,dx = 1$$

$$.95021 = \frac{1}{k}$$

$$1.0524 \approx k$$

$$\int_0^m 1.0524\, e^{-x}\,dx = \frac{1}{2}$$

$$-e^{-x}\Big|_0^m = \frac{1}{2(1.0524)}$$

$$-e^{-m} - (-1) = \frac{1}{2(1.0524)}$$

$$e^{-m} \approx .52489$$

$$m \approx .645$$

Let’s do this problem from a different textbook:

The random variable X has a probability density function defined by

$$f(x) = \begin{cases} k\left(-x^2+2x+15\right) & 0 \le x \le 5 \\ 0 & elsewhere \end{cases}$$

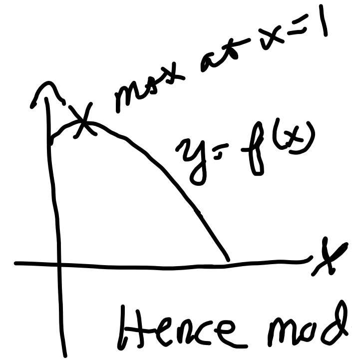
Find k.

Determine the mean and the mode

Find the value of median

$$k\int_0^5 (-x^2+2x+15)\,dx = 1$$

$$k = \frac{3}{175}$$



Hence mode = 1

$$\mu = \int_0^5 x f(x)\,dx$$

$$= \frac{55}{28}$$

$$\int_0^m f(x)\,dx = \frac{1}{2}$$

$$\frac{3}{175}\left[-\frac{x^3}{3}+x^2+15x\right]_0^m = \frac{1}{2}$$

$$X \approx 1.857$$

The lifetime Y, in tens of hours, of light globes produced by a certain company has a probability density function determine by

$$f(y) = \begin{cases} \frac{3}{500} y(10 - y), & 0 \le y \le 10 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the mean life of a light globe

(b) Find the median life of a light globe

$$\mu = \int_0^{10} y f(y)\, dy$$

$$= 5$$

BUT IN TENS OF HOURS

so $\mu = 50$ hours

$$\int_0^m \frac{3}{500} y(10-y)\, dy = \frac{1}{2}$$

$$5m^2 - \frac{m^3}{3} = \frac{250}{3}$$

$$m = 5$$

so median

50 HOURS

There is a nice summary at the bottom of page 749:

SUMMARY

Discrete random variable	Continuous random variable
• $\mu = \mathrm{E}(X) = \sum x p_x$	• $\mu = \mathrm{E}(X) = \int x f(x)\, dx$
• $\sigma^2 = \mathrm{Var}(X) = \mathrm{E}(X-\mu)^2$	• $\sigma^2 = \mathrm{Var}(X) = \mathrm{E}(X-\mu)^2$
$= \sum (x-\mu)^2 p_x$	$= \int (x-\mu)^2 f(x)\, dx$
$= \sum x^2 p_x - \mu^2$	$= \int x^2 f(x)\, dx - \mu^2$
$= \mathrm{E}(X^2) - \{\mathrm{E}(X)\}^2$	$= \mathrm{E}(X^2) - \{\mathrm{E}(X)\}^2$

Since we have time [and since I spent my own money], let's look at Mr. Sparrow's Power Point on Continuous Probability Density Functions