IBHL Review Two

Let $z = \cos\theta + i\sin\theta$, for $\frac{-\pi}{4} < \theta < \frac{\pi}{4}$

(a)(i) Find z^3 using the binomial theorem

$(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3\cos^2\theta\, i\sin\theta$
$+ 3\cos\theta\, i^2\sin^2\theta + i^3\sin^3\theta$
$= \cos^3\theta + 3\cos^2\theta\, i\sin\theta$
$- 3\cos\theta\sin^2\theta - i\sin^3\theta$

(ii) Use deMoivre's Theorem to show that

$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$

$\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$ and

$\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$

$\cos(3\theta) = 3\cos^3\theta - 3\cos\theta\sin^2\theta$
$= 3\cos^3\theta - 3\cos\theta(1-\cos^2\theta)$
$= 4\cos^3\theta - 3\cos\theta$

$\sin(3\theta) = 3\cos^2\theta\sin\theta - \sin^3\theta$
$= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta$
$= 3\sin\theta - 4\sin^3\theta$

(b) Hence, prove that $\dfrac{\sin 3\theta - \sin\theta}{\cos 3\theta + \cos\theta} = \tan\theta$

$$\frac{\sin 3\theta - \sin\theta}{\cos 3\theta + \cos\theta} = \frac{3\sin\theta - 4\sin^3\theta - \sin\theta}{4\cos^3\theta - 3\cos\theta + \cos\theta}$$

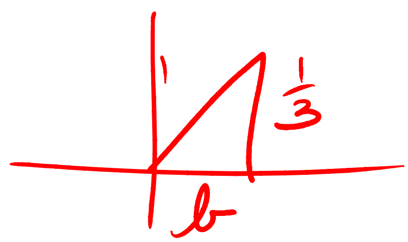
$$= \frac{\cancel{2}\sin\theta\,\cancel{(1-2\sin^2\theta)}}{\cancel{2}\cos\theta\,\cancel{(2\cos^2\theta - 1)}}$$

$$= \frac{\sin\theta}{\cos\theta}$$

(c) Given that $\sin\theta = \dfrac{1}{3}$, find the exact value of

$\tan 3\theta$

$$\sin\theta = \frac{1}{3}$$



$$\cos\theta = \frac{2\sqrt{2}}{3}$$

$$\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta}$$

$$= \frac{3\sin\theta - 4\sin^3\theta}{4\cos^3\theta - 3\cos\theta}$$

$$= \frac{23}{20}\sqrt{2}$$

Consider the system of equations

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ -42 \end{pmatrix} \text{where } T = \begin{pmatrix} -1 & 3 & 0 \\ 0 & 2 & r \\ 3r & 0 & s \end{pmatrix}$$

(a) Find the solution of the system when $r = 0,\ s = 3$

(b) The solution of the system is not unique

(i) Show that $s = \frac{9}{2}r^2$

(ii) When r = 2 and s = 18, show that the system can be solved, and find the general solution

(c) Use Mathematical Induction to prove that, when r = 0

$$T^n = \begin{pmatrix} (-1)^n & 2^n - (-1) & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & s^n \end{pmatrix} \text{ for n} \in \mathbb{Z}^+$$