IBHL Matrices

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Topic 4—Core: Matrices

4.3	Determinant of a 2×2 matrix	$A=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det A = ad-bc$
	Inverse of a 2×2 matrix	$A=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^{-1}=\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, $ad \neq bc$
	Determinant of a 3×3 matrix	$A=\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} \Rightarrow \det A = a\begin{vmatrix} e & f \\ h & k \end{vmatrix} - b\begin{vmatrix} d & f \\ g & k \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

An m by n matrix has m rows and n columns

Two matrices are ***equal*** if they have exactly the same shape [order] and elements in corresponding positions are equal.

We can add or subtract matrices of the same order [shape or dimension]. Simply add or subtract corresponding elements.

Multiples of matrices

If a scalar t is multiplied by a matrix A, the result is matrix $t\,A$ obtained by multiplying every element of A by t.

Zero Matrices

A zero matrix is a matrix in which all elements are zero. [Used to show that matrices have an identity element for addition.]

If A is a matrix of any order and O is the corresponding zero matrix, then $A+O=O+A=A$.

Negative Matrices

The negative matrix A, denoted $-A$ is actually just $-1A$.
[A matrix added to its negative matrix equals the zero matrix.]

$$A+(-A)=(-A)+A=O$$

Ordinary algebra	Matrix algebra
• If a and b are real numbers then so is ab.	• If $\mathbf{A}$ and $\mathbf{B}$ are matrices that can be multiplied then $\mathbf{AB}$ is also a matrix.
• $ab=ba$ for all a, b	• In general $\mathbf{AB} \neq \mathbf{BA}$.
• $a0=0a=0$ for all a	• If $\mathbf{O}$ is a zero matrix then $\mathbf{AO}=\mathbf{OA}=\mathbf{O}$ for all $\mathbf{A}$.
• $a(b+c)=ab+ac$	• $\mathbf{A(B+C)}=\mathbf{AB}+\mathbf{AC}$
• $a\times 1=1\times a=a$	• If $\mathbf{I}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ then $\mathbf{AI}=\mathbf{IA}=\mathbf{A}$ for all 2×2 matrices $\mathbf{A}$.
• a^n exists for all $a\geqslant 0$	• $\mathbf{A}^n$ for $n\geqslant 2$ can be determined provided that $\mathbf{A}$ is a square and n is an integer.

Determinant

If $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$, then $|A|=ad-bc$

If $|A|$ is equal to zero, then A is called a singular matrix.

Inverse of a 2×2 **Matrix** [You need to be able to do without your TI. Don't worry, you are given this in the formula packet]

If $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1}=\dfrac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

♪ A has an inverse provided its determinant $\neq 0$

♫ $A\,A^{-1}=A^{-1}A=I$

The 3×3 Determinant

Let A = $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

To find $|A|$ we can do a neat trick which involves covering up parts of the matrix. We will cover up the elements in row 1, but one at a time.

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

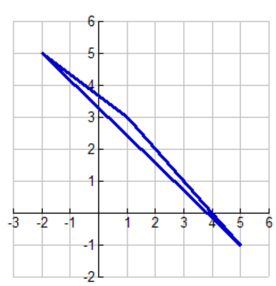
Cover the row and the column and find the determinant of what is left. Note: the middle term needs a subtraction sign.

Remember: if $|A| = 0$, then we have a singular matrix
Once again, the formula for the determinant is given in the official IB formula packet.

Most common use of matrices is to solve systems of equations. However, IB also includes matrices with their mathematical induction topic. Another interesting use of matrices is finding the area of a triangle whose vertices are given as points in a coordinate plane.

Example:
Find the area of a triangle ABC whose vertices are $A(1,3)$, $B(5,-1)$ and $C(-2,5)$
To find the area of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) just use this handy-dandy formula:

$$Area = \left| \tfrac{1}{2} |A| \right| \text{ where } A = \begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}$$



$$A = \begin{pmatrix} 1 & 3 & 1 \\ 5 & -1 & 1 \\ -2 & 5 & 1 \end{pmatrix}$$

Hence, the area of the triangle is

$Area = \left| \frac{1}{2} \bullet -4 \right| = 2$

If your area is equal to zero, then that means that the three points given are collinear.

Remember that your TI can do all sorts of matrix operations.

Here is a typical IBHL matrix problem.

Let $A = \begin{pmatrix} 2 & 6 \\ k & -1 \end{pmatrix}$ and $B = \begin{pmatrix} h & 3 \\ -3 & 7 \end{pmatrix}$ where h and $k \in Z$.

Given that $\det A = \det B$* and that $\det AB = 256h$

(a) Show that h satisfies the equation $49h^2 - 130h + 81 = 0$;

(b) Hence, find the value of k

* Big Clue

We have to start somewhere. Let's first find $\det A$ and $\det B$ then ponder about $\det AB$

$\det A = |A| = -2 - 6k$

$\det B = |B| = 7h + 9$

since $|A| = |B|$

then $-2 - 6k = 7h + 9$

$k = \dfrac{11 + 7h}{-6}$

$$(\det A)(\det B) = \det AB$$

$$\det AB = (\det B)^2$$

$$(\det B)^2 = (7h+9)^2 = 49h^2 + 126h + 81$$

$$256h = 49h^2 + 126h + 81$$

$$0 = 49h^2 - 130h + 81$$

Solve for h then use the value to find k

$$h = \frac{130 \pm \sqrt{(-130)^2 - 4(49)(81)}}{2(49)}$$

$$h = \frac{130 \pm 32}{98}$$

only use $h = 1$

$$k = \frac{-11 - 7(1)}{6} = -3$$ and $-3 \in \mathbb{Z}$

Solving systems of equations using row operations

$x - 3y - z = -1$

$2x - y + 4z = 3$

$3x + y + 9z = 2$

$$\left|\begin{array}{ccc|c} 1 & -3 & -1 & -1 \\ 2 & -1 & 4 & 3 \\ 3 & 1 & 9 & 2 \end{array}\right|$$

$2R1 - R2 = \text{NEW } R2$

$$\begin{array}{cccc|c} 2 & -6 & -2 & & -2 \\ 2 & -1 & 4 & & 3 \\ \hline 0 & -5 & -6 & & -5 \end{array}$$

$3R1 - R3 = \text{New } R3$

$$\begin{array}{cccc} 3 & -9 & -3 & -1 \\ 3 & 1 & 9 & 2 \\ \hline 0 & -10 & -12 & -3 \end{array}$$

$$\left|\begin{array}{ccc|c} 1 & -3 & -1 & -1 \\ 0 & -5 & -6 & -5 \\ 0 & -10 & -12 & -3 \end{array}\right|$$

$2R2 - R3 = \text{New } R3$

$$\begin{array}{cccc} 0 & -10 & -12 & -10 \\ 0 & -10 & -12 & -3 \\ \hline 0 & 0 & 0 & -7 \end{array}$$

$0x + 0y + 0z = -7$

NO SOLUTION

$$x - 2y + 3z = 2$$
$$4x - y + 7z = 6$$
$$3x + 8y - z = 2$$

$$\left|\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 4 & -1 & 7 & 6 \\ 3 & 8 & -1 & 2 \end{array}\right|$$

$4R1 - R2 = \text{New } R2$

$$\begin{array}{cccc} 4 & -8 & 12 & 8 \\ 4 & -1 & 7 & 6 \\ \hline 0 & -7 & 5 & 2 \end{array}$$

$3R1 - R3 = \text{NEW } R3$

$$\begin{array}{cccc} 3 & -6 & 9 & 6 \\ 3 & 8 & -1 & 2 \\ \hline 0 & -14 & 10 & 4 \end{array}$$

$$\left|\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & -7 & 5 & 2 \\ 0 & -14 & 10 & 4 \end{array}\right|$$

$2R2 - R3 = \text{New } R3$

$$\begin{array}{cccc} 0 & -14 & 10 & 4 \\ 0 & -14 & 10 & 4 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$0 = 0$

INFINITE # of SOLUTIONS

Introducing a parameter to find a general solution
[Do you remember this?]

Matrices with infinite solutions can have general solutions.

Give the general solutions for,

$4x + y + 4z = 7$

$2x - y + z = 2$

$-2x + 4y + z = 1$

Using row reduction from earlier we have established that these equations have infinite solutions.

Let $z = t$

$$\left|\begin{array}{ccc|c} 4 & 1 & 4 & 7 \\ 2 & -1 & 1 & 2 \\ -2 & 4 & 1 & 1 \end{array}\right|$$

after row reduction we are left with

$0x + 0y + 0z = 0$

$2x - y + t = 2$ (2nd line)

NOW WE CAN ISOLATE y

$$\boxed{y = 2x + t - 2}$$

Sub into 3rd line:

$$-2x + 4(2x + t - 2) + t = 1$$

$$5t + 6x - 9 = 0$$

$$\boxed{t = \frac{9 - 6x}{5}} \Rightarrow x = \frac{9 - 5t}{6}$$

Sub into 1st line:

$$4\left(\frac{9 - 5t}{6}\right) + y + 4t = 7$$

$$4\left(\frac{9 - 5t}{6}\right) + y + 4t = 7$$

$$36 - 20t + 6y + 24t - 42 = 0$$

$$2t + 3y - 3 = 0$$

$$\boxed{t = \frac{3 - 3y}{2}}$$

$$\frac{9-6x}{5} = \frac{3-3y}{2} = z$$

$$\boxed{t = \frac{9 - 6x}{5}}$$

$$\boxed{z = t}$$

General solution can be expressed as:

$$(t=)\frac{9-6x}{5}=\frac{3-3y}{2}=z$$

Find the general solution for,

$$x-2y+3z=2$$
$$4x-y+7z=6$$
$$3x+8y-z=2$$

Let $z=t$

$$\frac{10-7x}{11}=\frac{2+7y}{5}=z$$