IBHL MATHEMATICAL INDUCTION REVISION

MATHEMATICAL INDUCTION IS A METHOD FOR PROVING A MATHEMATICAL STATEMENT IS TRUE FOR EVERY NATURAL NUMBER. THERE ARE JUST A FEW BASIC STEPS. [Yes, you will show ALL of these steps!]

EXAMPLE ONE:

Show, using mathematical induction, that

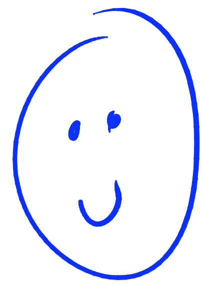
$$1+2+3+\ldots+n=\frac{n(n+1)}{2}$$

[Fancy-pants version] $\displaystyle\sum_{r=1}^{n} r=\frac{n(n+1)}{2}$

Step One: Write a statement to show what you are trying to prove.

Let $S(n)$ be the statement that for $n \in Z^+$

$$1+2+3+\ldots+n=\frac{n(n+1)}{2}$$



yay!

Step Two: Show that $S(1)$ is true

$$S(1)=1=\frac{1(1+1)}{2}$$

Step Three: Assume that the statement is true for $n=k$ and use that to show this implies that the statement is true for its successor, $n=k+1$

Assume that $S(k)$ is true for $k \in Z^+$, namely that

$1+2+3+\ldots+k=\dfrac{k(k+1)}{2}$. I need to show that

$$S(k+1)=\frac{(k+1)(k+1+1)}{2} \text{ or } S(k+1)=\frac{(k+1)(k+2)}{2}$$

In order to do this, we usually just have to scram up some "Algebra Magic" which uses our assumption about $S(k)$.

Note: You can only work on "one side" of the equation.

Step Four: "Algebra Magic"

$$S(k+1) = \underbrace{1+2+3+\ldots+k}_{\frac{k(k+1)}{2}} + k+1 = \frac{(k+1)(k+2)}{2}$$

$$\frac{k(k+1)}{2} + k+1 =$$

$$\frac{k(k+1) + 2(k+1)}{2} =$$

$$\frac{k^2 + k + 2k + 2}{2} =$$

$$\frac{k^2 + 3k + 2}{2} =$$

$$\frac{(k+1)(k+2)}{2} = \frac{(k+1)(k+2)}{2}$$

Step Five: Write your concluding statements.

Hence, since $S(1)$ is true and $S(k)$ implies that $S(k+1)$ is true for $k \in Z^+$, then by the principle of mathematical induction, $S(n)$ is true for $n \in Z^+$.

So, we just need those five basic steps!

Let's try another example.

Show that $n^3 + 2n$ is divisible by 3 for $n \in Z^+$

Step 1:

Let $S(n)$ be the statement that for $n \in Z^+$, $n^3 + 2n$ is DIVISIBLE by 3

Step Two: Show that $S(1)$ is true

$$S(1) = 1^3 + 2(1)$$
$$= 3$$

SO $S(1)$ is TRUE

Step Three: Your assumption step

Assume that $S(k)$ is true for $k \in Z^+$, namely that

$K^3 + 2K$ is DIVISIBLE by 3

I need to show that $S(K+1)$ is DIVISIBLE by 3

Step Four:

$$S(K+1) = (K+1)^3 + 2(K+1)$$
$$= K^3 + 3K^2 + 3K + 1 + 2K + 2$$
$$= \underbrace{K^3 + 2K} + 3K^2 + 3K + 3$$
$$= S(K) + 3(K^2 + K + 1)$$

SINCE $S(K)$ is DIVISIBLE by 3

Then $S(K+1)$ is ALSO DIVISIBLE by 3.

Step Five: Your concluding statements

Hence, since $S(1)$ is true and $S(k)$ implies that $S(k+1)$ is true for $k \in Z^+$, then by the principle of mathematical induction, $S(n)$ is true for $n \in Z^+$.

During the past six years, the IBHL exams have contained proofs by mathematical induction that have asked you to prove statements involving trigonometry, Calculus, complex numbers, matrices, and series.

Hint:
let $5^k - 1 = 4m$
$k, m \in \mathbb{Z}^+$

You try:

Show that $5^n - 1$ is divisible by 4 for all $n \in Z^+$

Let $S(n)$ be the statement that for $n \in \mathbb{Z}^+$, $5^n - 1$ is divisible by 4.

$S(1) = 5^1 - 1$
$= 4$ yay! $S(1)$ is TRUE

Assume for $k \in \mathbb{Z}^+$, $S(k)$ is TRUE
i.e. $5^k - 1$ is divisible by 4

Need to show $S(k+1)$ is divisible by 4.

$$
\begin{aligned}
S(k+1) &= 5^{k+1} - 1 \\
&= 5 \cdot 5^k - 1 \\
&= 5(5^k - 1) + 4 \\
&= 5(4m) + 4 \\
&= 4(5m + 1)
\end{aligned}
$$

$5^k - 1 = 4m$
$m, k \in \mathbb{Z}^+$

which is divisible by 4

Hence, since $S(1)$ is true and $S(k)$ implies that $S(k+1)$ is true for $k \in Z^+$, then by the principle of mathematical induction, $S(n)$ is true for $n \in Z^+$.

My IBHL Mathematical Induction Handout

Please do show ALL working on a separate piece of paper. You must clearly show all five steps. [No whining!]

I. Write an S(n) statement

II. Clearly show that S(1) is true

III. Write your assumption statement

IV. Show your mathematical work to prove that $S(k)$ implies $S(k+1)$

V. Write your concluding statements

[Mathematical Induction is covered in your textbook in chapter ten]

1. Show that $n^3 - n$ is divisible by 3 for $n \in Z^+$

2. Show that $1 + 3 + 5 + \ldots + (2n-1) = n^2$ for $n \in Z^+$

3. Show that $7^n - 1$ is divisible by 6 for $n \in Z^+$

4. Show that for
$$n \in Z^+ \; 1\times2\times3 + 2\times3\times4 + 3\times4\times5 + \ldots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$$

5. Show that $5^n + 3$ is divisible by 4 for $n \in Z^+$

Now for some problems from previous IBHL exams:

May 2007 Paper 2 TZ1

Prove by induction that $12^{n}+2\left(5^{n-1}\right)$ is a multiple of 7 for $n \in Z^{+}$

[Hint: For this one, you might want to call your multiples of 7 to be in the form of $7m$ for $m \in Z^{+}$]

May 2010 Paper 1 TZ1

(a) Show that $\sin 2nx = \sin((2n+1)x)\cos x - \cos((2n+1)x)\sin x$

(b) Hence prove, by induction that

$$\cos x + \cos 3x + \cos 5x + \ldots + \cos((2n-1)x) = \frac{\sin 2nx}{2x \sin x}$$

May 2005 Paper 1 TZ1

Using mathematical induction, prove that $\sum_{r=1}^{n}(r+1)2^{r-1} = n\left(2^{n}\right)$ for $n \in Z^{+}$

We will do more mathematical induction problems throughout the year.