MY IBHL COMPLEX NUMBER REVISION

Ibhlcomplexnotes.doc

Here are the important rules from our formula packet:

1.5	Complex numbers	$z = a + \mathrm{i}b = r(\cos\theta + \mathrm{i}\sin\theta) \;= re^{i\theta} = r\,\mathrm{cis}\,\theta$
1.7	De Moivre's theorem	$\left[r(\cos\theta + \mathrm{i}\sin\theta)\right]^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta) = r^n \mathrm{e}^{in\theta} \;= r^n\,\mathrm{cis}\,n\theta$

So that is all you have to use on the actual exam. However, there are a ton of rules for this topic, in other words, you need to make sure that you have a working knowledge of these rules. At least until May 9th.

There are two, count them two, chapters in our textbook devoted to complex numbers.

Some basics:

Any number of the form $a+bi$, $a,b \in R$, and $i=\sqrt{-1}$ is called a complex number in Cartesian form. [Remember that there are two other forms – polar and Euler]

$a+bi$ and $a-bi$ are called complex conjugates

If $z=a+bi$, then its conjugate is written as $z^*=a-b\,i$

Rules to cherish:

We can add, subtract, multiply and divide complex numbers in the same way that we perform these operations with radical expression – after all, i is a radical expression.

Re / Im

$$(a+bi)+(c+di)=(a+c)+(b+d)i \qquad \textbf{addition}$$

$$(a+bi)-(c+di)=(a-c)+(b-d)i \qquad \textbf{subtraction}$$

$$(a+bi)(c+di)=ac+adi+bci+bdi^2 \qquad \textbf{multiplication}$$

$$\frac{a+bi}{c+di}=\left(\frac{a+bi}{c+di}\right)\left(\frac{c-di}{c-di}\right)=\frac{ac-adi+bci-bdi^2}{c^2+d^2} \qquad \textbf{division}$$

clever form of one

Division is the tough operation.

Here is a division example from our textbook:

Example 7

If $z = 3 + 2i$ and $w = 4 - i$ find $\frac{z}{w}$ in the form $a + bi$, where a and b are real.

$$\frac{z}{w} = \frac{3+2i}{4-i}$$
$$= \left(\frac{3+2i}{4-i}\right)\left(\frac{4+i}{4+i}\right)$$
$$= \frac{12+3i+8i+2i^2}{16-i^2}$$
$$= \frac{10+11i}{17}$$
$$= \tfrac{10}{17} + \tfrac{11}{17}i$$

$w^* = 4+i$

clever form of one

Using those complex conjugates in simple algebraic ways

Note:

- Quadratics with real coefficients are called **real quadratics**. This does not necessarily mean that its zeros are real.
- If a quadratic equation has **rational coefficients** and an **irrational root** of the form $c + d\sqrt{n}$, then $c - d\sqrt{n}$ is also a root. These roots are **radical conjugates**.
- If a real quadratic equation has $\Delta < 0$ and $c + di$ is a complex root then $c - di$ is also a root. These roots are **complex conjugates**.

Theorem: If $c + di$ and $c - di$ are roots of a quadratic equation, then the quadratic equation is $a(x^2 - 2cx + (c^2 + d^2)) = 0$ for some constant $a \neq 0$.

These are from the online textbook

Example 3

Solve for x:

a $x^2+9=0$

b $x^3+2x=0$

a
$$x^2+9=0$$
$$\therefore \quad x^2-9i^2=0$$
$$(x+3i)(x-3i)=0$$
$$\therefore \quad x=\pm 3i$$

b
$$x^3+2x=0$$
$$\therefore \quad x(x^2+2)=0$$
$$\therefore \quad x(x^2-2i^2)=0$$
$$x(x+i\sqrt{2})(x-i\sqrt{2})=0$$
$$\therefore \quad x=0 \text{ or } \pm i\sqrt{2}$$

Example 4

Solve for x:
$x^2-4x+13=0$

$$x^2-4x+13=0 \quad \therefore \quad x=\frac{4\pm\sqrt{16-4(1)(13)}}{2}$$
$$\therefore \quad x=\frac{4\pm\sqrt{-36}}{2}$$
$$\therefore \quad x=\frac{4\pm 6i}{2}$$
$$\therefore \quad x=2+3i \quad \text{or} \quad 2-3i$$

Example 5

Solve for x:
$x^4+x^2=6$

$$x^4+x^2=6$$
$$\therefore \quad x^4+x^2-6=0$$
$$\therefore \quad (x^2+3)(x^2-2)=0$$
$$\therefore \quad (x+i\sqrt{3})(x-i\sqrt{3})(x+\sqrt{2})(x-\sqrt{2})=0$$
$$\therefore \quad x=\pm i\sqrt{3} \quad \text{or} \quad \pm\sqrt{2}$$

Example 11

Find exact values of a and b if $\sqrt{2}+i$ is a root of $x^2+ax+b=0$, $a, b \in \mathbb{R}$.

Since a and b are real, the quadratic has real coefficients

$\therefore$ $\sqrt{2}-i$ is also a root

$\therefore$ sum of roots $=\sqrt{2}+i+\sqrt{2}-i=2\sqrt{2}$

product of roots $=(\sqrt{2}+i)(\sqrt{2}-i)=2+1=3$

Thus $a=-2\sqrt{2}$ and $b=3$.

There is a proof in your textbook

Unless you have an eidetic memory, you should have written this example down.

Let's look at our Complex Numbers Operations handout

On your own, do numbers 1, 2, 3, 4, and 5

(5)

$$z = a + bi$$

$$z^2 = 5 - 12i$$

$$z^2 = (a+bi)^2$$

$$= a^2 + 2abi + b^2 i^2$$

$$= \underbrace{a^2 - b^2}_{Re} + \underbrace{2abi}_{Im}$$

$$5 = a^2 - b^2$$

$$-12 = 2ab \rightarrow b = -\frac{6}{a}$$

$$5 = a^2 - \left(-\frac{6}{a}\right)^2$$

$$5 = a^2 - \frac{36}{a^2}$$

$$a^2 - \frac{36}{a^2} - 5 = 0$$

$$a^4 - 36 - 5a^2 = 0$$

$$a^4 - 5a^2 - 36 = 0$$

$$(a^2+4)(a^2-9) = 0$$

$$a^2 + 4 = 0 \quad \text{or} \quad a^2 - 9 = 0$$

$$a^2 = -4 \qquad a = \pm 3$$

$$a = \pm\sqrt{-4} \qquad a \in \mathbb{R}$$

$$a = 3,\ b = -2$$

$$a = -3,\ b = 2$$

We’ll do the rest of the problems later

Let’s take a closer look at complex conjugates and their various and numerous properties

- $(z^*)^* = z$
- $(z_1 + z_2)^* = z_1^* + z_2^*$ and $(z_1 - z_2)^* = z_1^* - z_2^*$
- $(z_1 z_2)^* = z_1^* \times z_2^*$ and $\left(\frac{z_1}{z_2}\right)^* = \frac{z_1^*}{z_2^*}, \quad z_2 \neq 0$

- $(z^n)^* = (z^*)^n$ for integers $n = 1, 2$ and 3
- $z + z^*$ and zz^* are real.

Example 12

Show that $(z_1 + z_2)^* = z_1^* + z_2^*$ for all complex numbers z_1 and z_2.

Let $z_1 = a + bi$ and $z_2 = c + di$ $\therefore$ $z_1^* = a - bi$ and $z_2^* = c - di$

Now $z_1 + z_2 = (a + c) + (b + d)i$ $\therefore$ $(z_1 + z_2)^* = (a + c) - (b + d)i$

$$= a + c - bi - di$$
$$= a - bi + c - di$$
$$= z_1^* + z_2^*$$

- If z is any complex number then $z + z^*$ is real and zz^* is real.
- $(z^*)^* = z$
- If z_1 and z_2 are any complex numbers then
  - $(z_1 + z_2)^* = z_1^* + z_2^*$
  - $(z_1 - z_2)^* = z_1^* - z_2^*$
  - $(z_1 z_2)^* = z_1^* z_2^*$
  - $\left(\dfrac{z_1}{z_2}\right)^* = \dfrac{z_1^*}{z_2^*}$
- $(z^n)^* = (z^*)^n$ for all positive integers n
- $(z_1 + z_2 + z_3 + ... + z_n)^* = z_1^* + z_2^* + z_3^* + ... + z_n^*$

  and $(z_1 z_2 z_3 .... z_n)^* = z_1^* z_2^* z_3^* .... z_n^*$

I have tons of powerpoints

Modulus or magnitude of a complex number is just the length of the vector representation of the complex on the Argand diagram.

The argument is the angle that the vector representation makes with the x-axis.

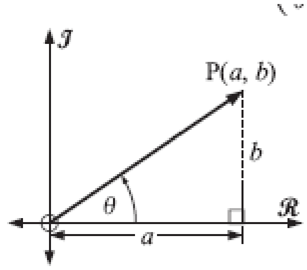
Converting to polar coordinates or $r\cos\theta + i\sin\theta$ form. [This is usually written as $rcis\theta$]

r is the modulus of the complex number and θ is the argument of the complex number

Some people like to draw the Argand diagram to find r and θ. Some people just use the following formulas:

$|\ a+ib\ |$ or the modulus = $\sqrt{a^2+b^2}$

$$\arg(a+ib) = \tan^{-1}\left(\frac{b}{a}\right)$$



Properties of argument

$$\arg(zw) = \arg z + \arg w$$

$$\arg(z^n) = n\arg z$$

$\arg\left(\frac{z}{w}\right) = \arg z - \arg w$ [Hmm, these look familiar]

More useful shortcuts:

$$[r,\ \theta]\times[s,\Phi]=[rs,\ \ \theta+\Phi]$$

$$[r,\theta]\div[s,\ \Phi]=\frac{r}{s},\ \ \theta-\Phi$$

Some problems to ponder:

Find the modulus and argument of the complex number $\dfrac{33-i}{10+3i}$

It might be easier if we get this into $a+ib$ form

Write $\frac{-1+i}{1+i\sqrt{3}}$ in the form $a+ib$

We just need to do the same procedure as the previous problem.

(b) Express $-1+i$ and $1+i\sqrt{3}$ in polar form, and hence the division in (a) in polar form as well.