# Calculus Potpourri Day One

## This one is non-calculator

(a) Show that $\sin 2nx = \sin((2n+1)x)\cos x - \cos((2n+1)x)\sin x$. *[2 marks]*

(b) **Hence** prove, by induction, that

$$\cos x + \cos 3x + \cos 5x + \ldots + \cos((2n-1)x) = \frac{\sin 2nx}{2\sin x},$$

for all $n \in \mathbb{Z}^+$, $\sin x \neq 0$. *[12 marks]*

(c) Solve the equation $\cos x + \cos 3x = \frac{1}{2}$, $0 < x < \pi$. *[6 marks]*

Wow! 12 marks for a little proof by induction. You are so going to do well! That being said, you have 10 minutes.

COMPOUND ANGLE IDENTITY

$\sin(A-B) = \sin A\cos B - \cos A\sin B$

$A = (2n+1)x$

$B = x$

$\sin((2n+1)x)\cos x - \cos((2n+1)x)\sin x$

$= \sin[(2n+1)x - x]$

$= \sin[2nx + x - x]$

$= \sin(2nx)$

ASSUME $S(k)$ is TRUE

NAMELY $\cos x + \cos 3x + \cos 5x + \ldots + \cos((2k-1)x) = \frac{\sin 2kx}{2\sin x}$

FOR $k \in \mathbb{Z}^+$

$S(k+1) = \cos x + \cos 3x + \dots + \cos((2k-1)x) + \cos((2k+1)x) = \dfrac{\sin 2(k+1)x}{2\sin x}$

We must show that $S(k+1) = \dfrac{\sin 2(k+1)x}{2\sin x}$

$$\underbrace{\cos x + \cos 3x + \dots + \cos((2k-1)x)} + \cos((2k+1)x) =$$

$$\frac{\sin 2kx}{2\sin x} + \cos((2k+1)x) =$$

$$\frac{\sin 2kx + 2\cos(2k+1)x \sin x}{2\sin x} =$$

$$\frac{\sin(2k+1)x\cos x - \cos(2k+1)x\sin x + 2\cos(2k+1)x\sin x}{2\sin x} =$$

$$\frac{\sin(2k+1)x\cos x + \cos(2k+1)x\sin x}{2\sin x} =$$

$$\frac{\sin(2k+2)x}{2\sin x} =$$

$$\frac{\sin 2(k+1)x}{2\sin x} = \qquad \text{TA-DA!}$$

$$\cos x + \cos 3x = \frac{1}{2} \qquad 0 < x < \pi$$

$$\frac{\sin 4x}{2\sin x} = \frac{1}{2}$$

$$\cancel{2}\sin 4x = \cancel{2}\sin x$$

$$4x = \pi - x \rightarrow x = \frac{\pi}{5}$$

$$4x = 2\pi - x \rightarrow x = \frac{2\pi}{5}$$

$$4x = 3\pi - x \rightarrow x = \frac{3\pi}{5}$$

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# Another easy paper one problem [5 minutes]

The function $f$ is defined by $f(x) = e^{x^2-2x-1.5}$.

$e^{x^2-2x-1,5}$

(a) Find $f'(x)$. *[2 marks]*

(b) You are given that $y = \frac{f(x)}{x-1}$ has a local minimum at $x = a$, $a > 1$. Find the value of $a$. *[6 marks]*

$f'(x) = (2x-2)e^{x^2-2x-1,5}$

$$y' = \frac{(x-1)(2x-2)\left(e^{x^2-2x-1.5}\right) - \left(e^{x^2-2x-1.5}\right)}{(x-1)^2}$$

let $y' = 0$

$$0 = \frac{e^{x^2-2x-1.5}\left(2(x-1)^2-1\right)}{(x-1)^2}$$

$$0 = \frac{e^{x^2-2x-1.5}\left(2x^2-4x+1\right)}{(x-1)^2}$$

$a > 1$ so $a = 1 + \sqrt{\frac{1}{2}}$

# Another 5 minute one [non-calculator]

Find the value of $\int_0^1 t\ln(t+1)\,dt$.

$$u = \ln(t+1) \quad v = \frac{t^2}{2}$$
$$du = \frac{1}{t+1}dt \quad dv = t\,dt$$
$$uv - \int v\,du$$

$$= \frac{t^2}{2}\ln(t+1)\Big|_0^1 - \int_0^1 \frac{t^2}{2}\left(\frac{1}{t+1}\right)dt$$

$$= \frac{t^2}{2}\ln(t+1)\Big|_0^1 - \frac{1}{2}\int_0^1 \frac{t^2}{t+1}\,dt$$

$$= \frac{t^2}{2}\ln(t+1)\Big|_0^1 - \frac{1}{2}\int_0^1 \left(t - 1 + \frac{1}{1+t}\right)dt$$

$$= \frac{t^2}{2}\ln(t+1)\Big|_0^1 - \frac{1}{2}\left[\frac{t^2}{2} - t + \ln(t+1)\right]\Big|_0^1$$

$$= \frac{1}{4}$$

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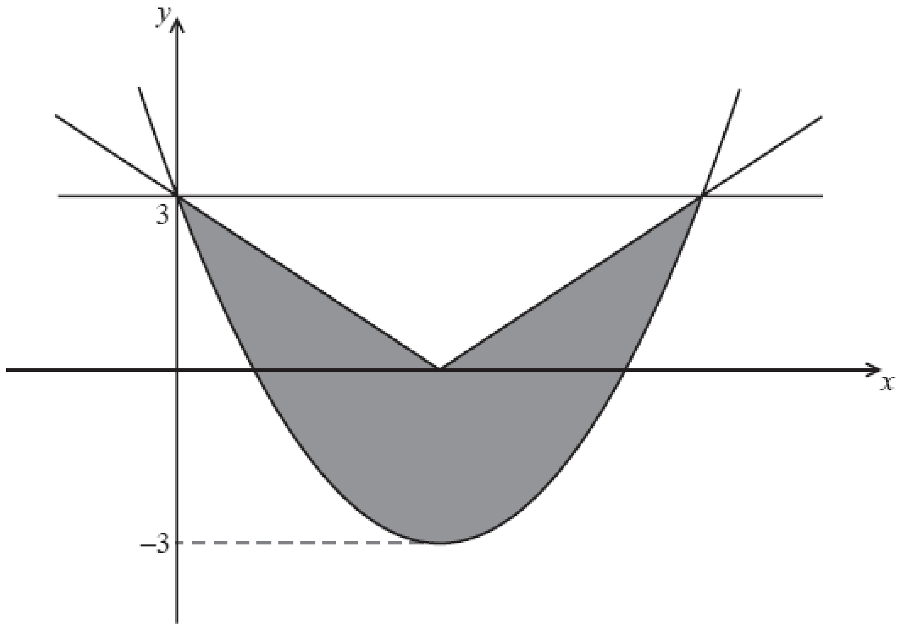
This one is calculator

The diagram below shows the graphs of $y = \left| \frac{3}{2}x - 3 \right|$, $y = 3$ and a quadratic function, that all intersect in the same two points.




Given that the minimum value of the quadratic function is $-3$, find an expression for the area of the shaded region in the form $\int_0^t (ax^2 + bx + c)\,dx$, where the constants

$a$, $b$, $c$ and $t$ are to be determined. (Note: The integral does not need to be evaluated.)

# A big calculator question

A body is moving through a liquid so that its acceleration can be expressed as

$$\left(-\frac{v^2}{200}-32\right)\text{m s}^{-2},$$

where $v\ \text{m s}^{-1}$ is the velocity of the body at time $t$ seconds.

The initial velocity of the body was known to be $40\ \text{m s}^{-1}$.

(a) Show that the time taken, $T$ seconds, for the body to slow to $V\ \text{m s}^{-1}$ is given by

$$T = 200\int_V^{40}\frac{1}{v^2+80^2}\,\mathrm{d}v\,.$$ *[4 marks]*

(b) (i) Explain why acceleration can be expressed as $v\dfrac{\mathrm{d}v}{\mathrm{d}s}$, where $s$ is displacement, in metres, of the body at time $t$ seconds.

(ii) **Hence** find a similar integral to that shown in part (a) for the distance, $S$ metres, travelled as the body slows to $V\ \text{m s}^{-1}$. *[7 marks]*

(c) **Hence**, using parts (a) and (b), find the distance travelled and the time taken until the body momentarily comes to rest. *[3 marks]*