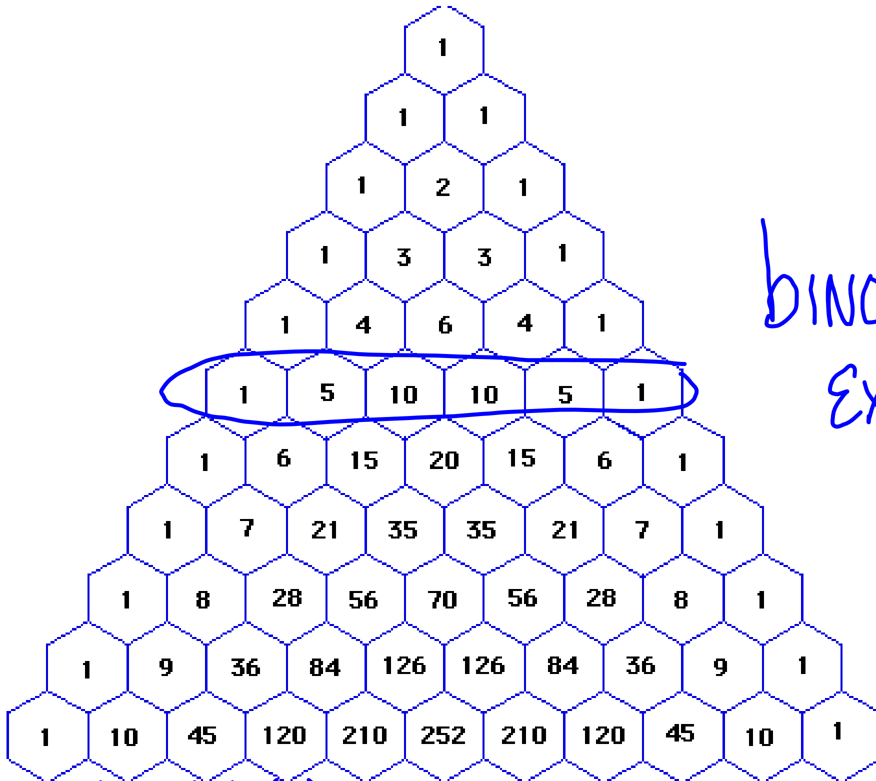
Binomial Theorem

Remember Pascal's Triangle?



BINOMIAL EXPANSION

$\binom{10}{0}\binom{10}{1}\binom{10}{2}\binom{10}{3}\binom{10}{4}\binom{10}{5}$ . . . . .

From: Google Images

One of the many amazing mathematical applications of Pascal's Triangle is the expansion of $(a+b)^n$ for $n = 0, 1, 2, \ldots$

For example:
$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

So what would be the expansion of $(a+b)^5$?

$a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3$
$+ 5ab^4 + b^5$

We can use this method for any binomial.

Expand: $(2x+5)^4$ Let $a = 2x$, $b = 5$

$a^4 + 4a^3b + 6a^2b^2$
$+ 4ab^3 + b^4$

$= (2x)^4 + 4(2x)^3(5)$
$+ 6(2x)^2(5)^2 + 4(2x)(5)^3 + (5)^4$
$= 16x^4 + 160x^3 + 600x^2 + 1000x + 625$

If you do not want to write out Pascal's Triangle to find the necessary coefficients of your binomial expansion, then you can use your handy-dandy g.d.c. to find them!

Since each element of Pascal's Triangle also can be thought of as n choose r [or $\begin{pmatrix} n \\ r \end{pmatrix}$, then we can use this feature on our g.d.c. to find these numbers.

$$\binom{0}{0}$$

$$\binom{1}{0}\binom{1}{1}$$

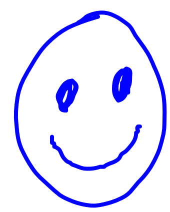
$$\binom{2}{0}\binom{2}{1}\binom{2}{2}$$

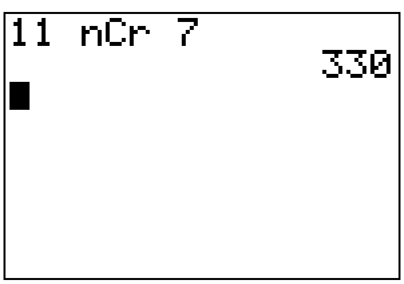
$$\binom{3}{0}\binom{3}{1}\binom{3}{2}\binom{3}{3}$$

Let's try this out! Find the coefficient of the x^7 term in the expansion of $(x+1)^{11}$

The binomial coefficient will be $\binom{11}{7}$

$330x^7$





So, the x^7 term will be $330x^7$.

What would the x^7 term be in the expansion of $(x+3)^{11}$?
[Let $a = x,\ b = 3$]

$\binom{11}{7}(a)^7(b)^4$

$= 330\ x^7(81) = 26730\ x$

Find the ninth term of $\left(x-\frac{2}{x}\right)^{17}$

$a = x$

$b = -\frac{2}{x}$

$\binom{17}{8}(a)^9(b)^8$

$= 24310\, x^9\left(-\frac{2}{x}\right)^8$

$= 6223360\, x$

Find the coefficient of x^7 in the expansion of $\left(x^2+\frac{4}{x}\right)^{11}$

In this case, $a = x^2,\ b = \frac{4}{x}$. This is trickier because we have to be aware of the possible products of $a^{11-r}\, b^r$. As you can see from our textbook

$T_{r+1} = \binom{11}{r}\left(x^2\right)^{11-r}\left(4x^{-1}\right)^r$ which can be simplified to

$T_{r+1} = \binom{11}{r}\left(x^{22-2r}\right)\left(4^r x^{-r}\right)$

$T_{r+1} = \binom{11}{r}4^r x^{22-3r}$ and we need $22-3r = 7$

Hence, $r = 5$.

So, $T_6 = \binom{11}{5}(x^2)^{11-5}(4x^{-1})^5$

$$= 473088x^7$$

The coefficient of the x^7 term is 473088.

Of course this would be on the calculator-friendly portion of any text/exam!

Expand the following:

(a) $(x+2y)^4$

$= x^4 + 8x^3y + 24x^2y^2 + 32xy^3 + 16y^4$

(b) $\left(x^2 - \dfrac{1}{x}\right)^4$

$= x^8 - 4x^5 + 6x^2 - 4x^{-1} + x^{-4}$

(c) $\left(\dfrac{x}{2} + \dfrac{2}{x}\right)^6$

$= \dfrac{x^6}{64} + \dfrac{3}{8}x^4 + \dfrac{15}{4}x^2 + 20$

$+ 60x^{-2} + 96x^{-4} + 64x^{-6}$

Now let's consider this example

Example 6

Find the coefficient of x^5 in the expansion of $(x+3)(2x-1)^6$.

$$\begin{aligned}
&(x+3)(2x-1)^6 \\
&= (x+3)[(2x)^6 + \tbinom{6}{1}(2x)^5(-1) + \tbinom{6}{2}(2x)^4(-1)^2 + \ldots.] \\
&= (x+3)(2^6x^6 - \tbinom{6}{1}2^5x^5 + \tbinom{6}{2}2^4x^4 - \ldots.)
\end{aligned}$$

So terms containing x^5 are $\binom{6}{2}2^4x^5$ from ① and

$-3\binom{6}{1}2^5x^5$ from ②

$\therefore$ the coefficient of x^5 is $\binom{6}{2}2^4 - 3\binom{6}{1}2^5$

$= -336$

Find the specified term in each of the given binomial expressions.

Expression	Term
$(x+3)^6$	x^3 $\binom{6}{3}(x^3)(3^3)$
$(3x^2+1)^{10}$	x^{10} $\binom{10}{5}(3x^2)^5(1)^5$

$(2x-3y)^7$	x^2y^5	$\binom{7}{5}(2x)^2(-3y)^5$
$\left(7x-\dfrac{11}{x}\right)^7$	x	$\binom{7}{4}(7x)^4(-11x^{-1})^3$
$\left(2x-\dfrac{3}{p}\right)^8$	$\dfrac{x^2}{p^6}$	$\binom{8}{2}(2x)^2\left(-\dfrac{3}{p}\right)^6$

In one of the terms $(x^2-4y^3)^5$, the powers of x and y will be identical. Find this term, stating the coefficient and showing clearly all your working.

$a=x^2$ $\quad b=-4y^3$ $\quad (a+b)^5$

Find the term independent of x in the expansion of,

$$\left(3x^2-\frac{2}{x^3}\right)^5.$$

a) Find the first three terms in the expansion, in ascending powers of x for the expansion of $(2-x)^5$.

b) Find the value of the constant a for which the coefficient of x^4 in the expansion $(1+ax)(2-x)^5$ is 2.