Natural logarithms [ln]

We have used the "log" key on our TI which is really $\log_{10}$. The other logarithm key on our TI is the "ln" key which is really $\log_e$ where $e \approx 2.7183$. It is called the "natural" logarithm because the number e occurs in nature! But most people use e or ln in the banking world.

Let's look at Investigation 1 on page 104

INVESTIGATION 1 — ***e* OCCURS NATURALLY**

Suppose $\$u_1$ is invested at a fixed rate of 10% p.a. for 10 years.

If *one* interest payment is made each year, then using $u_{n+1} = u_1 r^n$ where r is the rate per period and n is the number of periods, the investment will be worth $\$u_{11}$ after 10 years and $u_{11} = u_1(1.1)^{10} \doteqdot u_1 \times 2.593\,742$ {we are multiplying by 1.1 which equals 110%}

If 10 interest payments are made each year, then $u_{11} = u_1(1.01)^{100} \doteqdot u_1 \times 2.704\,814$

What to do:

1. Calculate u_{11} for
   - **a** 100 interest payments per year {$r = 1.001$}
   - **b** 1000 interest payments per year
   - **c** 10 000 interest payments per year
   - **d** 100 000 interest payments per year
   - **e** 1 000 000 interest payments per year
2. If 1 000 000 interest payments are made each year, how frequently does this occur?
3. Comment on your results to **1** above

$u_{n+1} = u_1(r)^n$

Where r is the rate per period and n is the number of periods

We saw that $u_{11} = u_1(1.1)^{10}$ when $n = 10$

If ten interest payments are made per year [for 10years], then $r = 1.01$ 10% divided by ten = 1%

$$u_{11} = u_1(1.01)^{10 \bullet 10}$$

$$u_{11} = u_1(1.01)^{100}$$

$$(1.01)^{100} \approx 2.701814$$

If 100 interest payments were made per year, then $r = 1.001$

$$u_{11} = u_1(1.001)^{10 \bullet 100}$$

$$u_{11} = u_1(1.001)^{1000}$$

$$(1.001)^{1000} \approx 2.716923932$$

If 1000 interest payments were made per year, then $r = 1.0001$

$$u_{11} = u_1(1.0001)^{10 \bullet 1000}$$

$$u_{11} = u_1(1.0001)^{10000}$$

$$(1.0001)^{10000} \approx 2.718145927$$

Hmmm! Looks pretty close to e.

If 10000 interest payments were made per year, then $r = 1.00001$

$$u_{11} = u_1(1.00001)^{10 \bullet 10000}$$

$$(1.00001)^{100000} \approx 2.718268237$$

Looks like as the number of payments gets larger, the value of r gets close to the value of e and that's why we have the $A = Pe^{rt}$ formula for compound interest in banking!

From our textbook, page 106:

For continuous growth, we have shown that:

$u_n = u_0 e^{rt}$ where u_0 is the initial amount
r is the annual percentage rate
t is the number of years

where $u_0 = P$

Let's consider the graphs of the functions of the form:

$f(x) = ae^x$ where $a \in R$eals

Graph:

$y_1 = 2e^x$ $y(0) = 2$

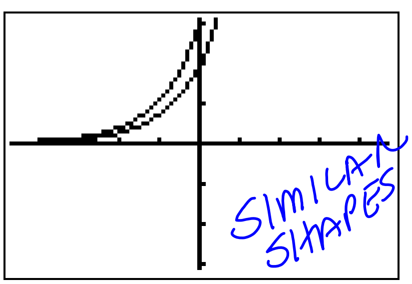
$y_2 = 3e^x$ $y(0) = 3$

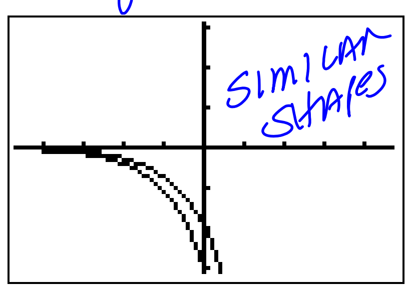
What do these graphs have in common?

$y_3 = -2e^x$ $y(0) = -2$

$y_4 = -3e^x$ $y(0) = -3$

$y = ae^x$
$y(0) = a$





$\lim_{x \to \infty} 2e^x = \infty$

$\lim_{x \to \infty} 3e^x = \infty$

growing

$\lim_{x \to -\infty} 2e^x = 0$

$\lim_{x \to -\infty} 3e^x = 0$

REFLECTIONS OF y_1, y_2

$\lim_{x \to -\infty} -2e^x = 0$

$\lim_{x \to -\infty} -3e^x = 0$

Now let's graph:

$y_1 = 2e^{2x}$

$y_2 = 2e^{3x}$

$y_3 = 2e^{4x}$

when b increases rate of change increases

$y = ae^{bx}$

$\lim_{x \to -\infty} 2e^{2x} = 0$

$\lim_{x \to -\infty} ae^{bx} = 0$

$a, b \in \mathbb{R}$

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WINDOW
 Xmin=-1.098612…
 Xmax=1.0986122…
 Xscl=1
 Ymin=-1
 Ymax=10
 Yscl=1
 Xres=1
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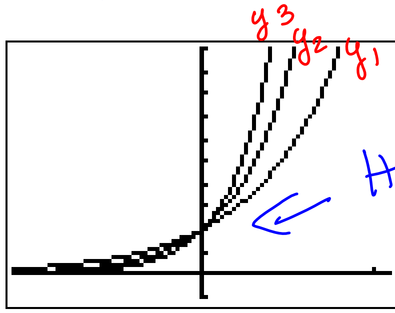


Hey! SAME intercept (0, 2)

What do you notice?

RATES of change

$\frac{d}{dx} 2e^{2x} = 4e^{2x}$

$\frac{d}{dx} 2e^{3x} = 6e^{3x}$

$\frac{d}{dx} 2e^{4x} = 8e^{4x}$

Fun Facts about the function $y = \ln x$!

If $f(x) = \ln x$, then $f^{-1}(x) = e^x$.

$\ln e^x = x$

The Laws of Natural Logarithms are located on page 110 of our textbook. [You should already know these!]

To solve an exponential equation of the form $e^x = a$, we just have to take the natural logarithm of both sides of the equation.

You can see this on page 112, Example 7

Example 7

Solve for x, giving your answer correct to 4 significant figures:

a $e^x = 30$ **b** $e^{\frac{x}{3}} = 21.879$ **c** $20e^{4x} = 0.0382$

a
$$e^x = 30$$
$$\therefore \quad x = \ln 30$$
$$\therefore \quad x \doteqdot 3.401$$

b
$$e^{\frac{x}{3}} = 21.879$$
$$\therefore \quad \frac{x}{3} = \ln 21.879$$
$$\therefore \quad \frac{x}{3} \doteqdot 3.0855.....$$
$$\therefore \quad x \doteqdot 9.257$$

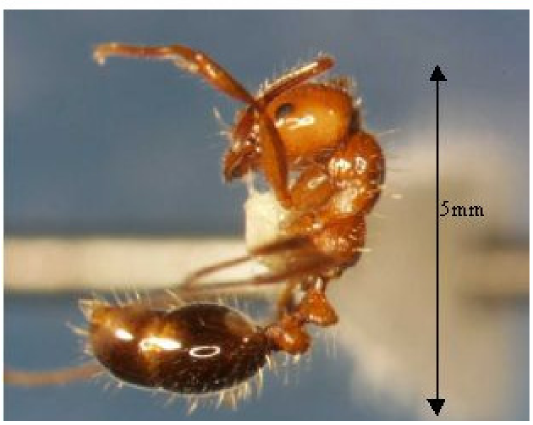
c
$$20e^{4x} = 0.0382$$
$$\therefore \quad e^{4x} = 0.00191$$
$$\therefore \quad 4x = \ln 0.00191$$
$$\therefore \quad 4x \doteqdot -6.2607....$$
$$\therefore \quad x \doteqdot -1.565$$

Let's look at Example 8 on page 113

From: http://www2.dpi.qld.gov.au/fireants/

Fire Ants - What are they?

Fire Ants, *Solenopsis invicta*, are serious pests which have been detected in Queensland, Australia. They inflict a painful sting and if not eradicated will seriously affect our lifestyle. They are the greatest ecological threat to Australia since the introduction of the rabbit and are potentially worse than the cane toad.



RASE the Restricted Area Search Engine

Find the answer fast

See maps of restricted areas in the Restricted Area Search Engine (RASE)

Example 8

A biologist, monitoring a fire ant infestation, notices that the area affected by the ants is given by $A_n = 1000 \times e^{0.7n}$ hectares, where n is the number of weeks after the initial observation.

a Draw an accurate graph of A_n against n and use your graph to estimate the time taken for the infested area to reach 5000 ha.

b Find the answer to **a** using logarithms.

c Check your answer to **b** using suitable technology.

$$5000 = 1000e^{.7x}$$

$$5 = e^{.7x}$$

$$\ln 5 = \ln e^{.7x}$$

$$\frac{\ln 5}{.7} = \frac{.7x}{.7}$$

$$\frac{\ln 5}{.7} = x$$

$$2.299^{\text{weeks}} \approx x$$

Homework: pages 116 and 117 –do all of review 5A