Series

From: http://www2.guhsd.net/algebra2/

When the terms of a sequence are added, the resulting expression is a **series**. A series can be finite or infinite.

FINITE SEQUENCE	INFINITE SEQUENCE
3, 6, 9, 12, 15	3, 6, 9, 12, 15, . . .
FINITE SERIES	**INFINITE SERIES**
3 + 6 + 9 + 12 + 15	3 + 6 + 9 + 12 + 15 + . . .

The sum of the first n terms of a sequence is:

$S_n = u_1 + u_2 + u_3 + ... + u_n$

For example:

Find the sum of the first five terms of the series $\{3n\}$ for $n = 1, 2, 3, 4, 5$

Our arithmetic sequence is: 3, 6, 9, 12, 15.

Our arithmetic series is:

$S_5 = 3 + 6 + 9 + 12 + 15$

$S_5 = 45$

An arithmetic series is the addition of successive terms of an arithmetic sequence.

The Famous Gauss Series Story

[Gauss is a famous mathematician! (1777-1855)]

The story told is that when Gauss was about 10 years old, he was disruptive in class. He cheesed off his teacher, so the teacher told him to sum up the first 100 numbers. Namely, 1+2+3+…+100. In less than a minute, Gauss blurted out the correct answer. [5050]

How did he do it?

$$S_{100} = 1 + 2 + 3 + 4 + \dots + 97 + 98 + 99 + 100$$
$$S_{100} = 100 + 99 + 98 + 97 + \dots + 4 + 3 + 2 + 1$$
$$2S_{100} = 101 + 101 + 101 + 101 + \dots + 101 + 101 + 101 + 101$$
$$2S_{100} = 100(101)$$
$$S_{100} = 50(101) = 5050$$

We can use a similar trick to sum any arithmetic series.

The terms of any arithmetic sequence can be written as:

$u_1, u_1+d, u_1+2d, u_1+3d, u_1+4d, \dots$

To find the sum of the first n terms [S_n]we would add the following:

$$u_1 + (u_1 + d) + (u_1 + 2d) + (u_1 + 3d) + \dots + (u_n - 2d) + (u_n - d) + u_n$$
$$u_1 + u_2 + u_3 + u_4 + \dots$$

Let’s do so rearranging [like Gauss did]:

$$S_n = u_n + (u_n - 2d) + (u_n - d) + \dots + (u_1 + 2d) + (u_1 + d) + u_1$$

Now add the two sums together and solve for S_n

$$2S_n = n\,(u_1 + u_n)$$

$$S_n = \frac{n}{2}(u_1 + u_n)$$

Let's see if this works for the Gauss problem.

$S_n = 1 + 2 + 3 + \ldots + 98 + 99 + 100$

In this problem, $n = 100$, $u_1 = 1$, $u_{100} = 100$

$$S_{100} = \frac{100}{2}(1 + 100)$$

$$S_{100} = 50(101)$$

$$S_{100} = 5050$$

Picture from Wikipedia

Johann Carl Friedrich Gauss

Carl Friedrich Gauss, painted by Christian Albrecht Jensen

Gauss, what a guy!

Note: If you don't know the last term of the sequence, then you can always use our handy formula $u_n = u_1 + (n-1)d$

So, an alternate formula for an arithmetic series would be:

$$S_n = \frac{n}{2}(u_1 + u_n)$$

now substitute $u_n = u_1 + (n-1)d$

$$S_n = \frac{n}{2}\left(u_1 + u_1 + (n-1)d\right)$$

$$S_n = \frac{n}{2}\left(2u_1 + (n-1)d\right)$$

$$S_n = \frac{n}{2}\left(2u_1 + (n-1)d\right)$$

Let's try some problems!

(1) Find the number of terms in the following series and then find the sum

1+3+5+7+…+99

This is an arithmetic series with $u_1 = 1$ and $u_n = 99$

The value of $d = 2$

Let's find n first:

$$u_n = 99$$

$$99 = u_1 + (n-1)d$$

$$99 = 1 + (n-1)(2)$$

$$99 = 1 + 2n - 2$$

$$99 = 2n - 1$$

$$n = 50$$

n is the number of terms

Now we can find the sum.

We can use

$$S_n = \frac{n}{2}(u_1 + u_n)$$

$$S_{50} = \frac{50}{2}(1+99)$$

$$S_{50} = 25(100)$$

$$S_{50} = 2500$$

(2) Find the sum of 8+11+14+17+… to 50 terms

This is an arithmetic series with $u_1 = 8$ and $d = 3$
We know that $n = 50$, so we can either find u_{50} or just use our alternate formula.

Let's try our alternate formula!

$$S_n = \frac{n}{2}\left(2u_1 + (n-1)d\right)$$
$$S_{50} = \frac{50}{2}\left(2(8) + (50-1)(3)\right)$$
$$S_{50} = 25\left(16 + (49)(3)\right)$$
$$S_{50} = 4075$$

(3) An arithmetic sequence has a common difference of 7 and $u_{11} = 53$. Find S_{19}.

We'll need to find u_1 first because it is necessary in either formula.
We can use: $u_n = u_1 + (n-1)d$

$$u_{11} = u_1 + (11-1)d$$
$$53 = u_1 + (10)(7)$$
$$-17 = u_1$$

Now we can use $S_n = \frac{n}{2}\left(2u_1 + (n-1)d\right)$ to find S_{19}

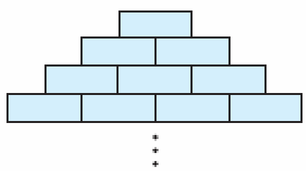
$u_1 \quad n \quad d$

$$S_{19} = \frac{19}{2}\left(2(-17) + (19-1)(7)\right)$$

$$S_{19} = 874$$

Let's take a look at #5 on page 53 of our textbook.

5



A bricklayer builds a triangular wall with layers of bricks as shown. If the bricklayer uses 171 bricks, how many layers are placed?

We know that $S_n = 171$, $d = 1$, and $u_1 = 1$. We need to find n.

$$S_n = \frac{n}{2}\left(2u_1 + (n-1)d\right)$$

$$171 = \frac{n}{2}\left(2(1) + (n-1)(1)\right)$$

$$171 = \frac{n}{2}(1+n)$$

$$342 = n(1+n)$$

$$342 = n^2 + n$$

$n^2 + n - 342 = 0$ Now we can solve for n.

$$(n+18)(n-17) = 0$$

Since we can not have negative rows, then $n = 17$

And now for something slightly different, let's consider page 53 #8.

Prove that the sum of the first n positive integers is $\frac{n(n+1)}{2}$ or show that $1+2+3+...+n=\frac{n(n+1)}{2}$

We know that $d=1$, $u_1=1$, and that there are n integers

$$S_n=\frac{n}{2}\left(2u_1+(n-1)d\right)$$

$$S_n=\frac{n}{2}\left(2(1)+(n-1)(1)\right)$$

$$S_n=\frac{n}{2}(2+n-1)$$

$$S_n=\frac{n}{2}(n+1)$$

$$S_n=\frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k=\frac{n(n+1)}{2}$$

SUMMATION NOTATION

Homework: page 53 #2, 3, 6, 7, 9, 11

2 Find the sum of:

- **a** $5 + 8 + 11 + 14 + ... + 101$
- **b** $50 + 49\frac{1}{2} + 49 + 48\frac{1}{2} + ... + (-20)$
- **c** $8 + 10\frac{1}{2} + 13 + 15\frac{1}{2} + ... + 83$

3 An arithmetic series has seven terms. The first term is 5 and the last term is 53. Find the sum of the series.

6 Each section of a soccer stadium has 44 rows with 22 seats in the first row, 23 in the second row, 24 in the third row, and so on. How many seats are there

- **a** in row 44
- **b** in a section
- **c** at a stadium which has 25 sections?

7 Find the sum of:

- **a** the first 50 multiples of 11
- **b** the multiples of 7 between 0 and 1000
- **c** the integers between 1 and 100 which are not divisible by 3.

9 Consider the series of odd numbers $1 + 3 + 5 + 7 + ...$

- **a** What is the nth odd number, that is, u_n?
- **b** Prove that "the sum of the first n odd numbers is n^2".
- **c** Check your answer to **b** by finding S_1, S_2, S_3 and S_4.

11 Three consecutive terms of an arithmetic sequence have a sum of 12 and a product of -80. Find the terms. (**Hint:** Let the terms be $x - d$, x and $x + d$.)