More Matrix Fun!

Results from 14G [our textbook page 326]

Ordinary algebra	Matrix algebra
• If a and b are real numbers then so is ab.	• If **A** and **B** are matrices that can be multiplied then **AB** is also a matrix.
• $ab = ba$ for all a, b	• In general $\mathbf{AB} \neq \mathbf{BA}$.
• $a0 = 0a = 0$ for all a	• If **O** is a zero matrix then $\mathbf{AO} = \mathbf{OA} = \mathbf{O}$ for all **A**.
• $a(b + c) = ab + ac$	• $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$
• $a \times 1 = 1 \times a = a$	• If $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$ for all 2×2 matrices **A**.
• a^n exists for all $a \geqslant 0$	• $\mathbf{A}^n$ for $n \geqslant 2$ can be determined provided that **A** is a square and n is an integer.

Let's work on page 327 #8 [Order matters in multiplication]

All matrices are 2 X 2 and I is the identity matrix

(a) $A(A+I)$

(b) $(B+2I)B$

(c) $A\left(A^2 - 2A + I\right)$

(d) $A\left(A^2 + A - 2I\right)$

(e) $\left(A + B\right)\left(C + D\right)$

(f) $\left(A + B\right)^2$

(g) $(A+B)(A-B)$

(h) $(3I-B)^2$

Let consider Example 8 on page 327
If $A^2 = 2A + 3I$, find A^3 and A^4 in the form $kA + lI$ where k and l are scalars.

$$
\begin{aligned}
A^3 &= A \times A^2 \\
&= A(2A+3I) \\
&= 2A^2 + 3AI \\
&= 2(2A+3I) + 3AI \\
&= 4A + 6I + 3AI \\
&= 4A + 6I + 3A \\
&= 7A + 6I
\end{aligned}
$$

$A^4 = A \times A^3$ Use the previous results

$= A(7A + 6I)$

Use previous results again

$= 7A^2 + 6AI$

$= 7(2a + 3I) + 6AI$

$= 14A + 21I + 6A$

$= 20A + 21I$

Now try #9 and 10 on page 327

Let's do #11 a, b, and c together:

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ find AB

$$AB = BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence, if $AB = 0$, then either $A = 0$ or $B = 0$ is not always true for matrices.

Now let $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ and find A^2 [How weird is this?]

$$A^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = A^3 = A^4$$

$$A^2 = A$$

(c) Can you spot the false statement?

$$A^2 = A \quad \therefore \quad A^2 - A = 0$$

$$A(A - I) = 0$$

$$\therefore A = 0$$

$$\text{or } A - I = 0$$

$$\therefore A = 0 \text{ or } A = I$$

In this case: $A - I \neq 0$, $A \neq 0$

ZERO PRODUCT PROPERTY DOES NOT HOLD IN THIS CASE

If time, consider Example 9 on page 328

Example 9

Find constants a and b such that $\mathbf{A}^2 = a\mathbf{A} + b\mathbf{I}$ for $\mathbf{A}$ equal to $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Since $\mathbf{A}^2 = a\mathbf{A} + b\mathbf{I}$,

$$\therefore \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = a\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + b\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} 1+6 & 2+8 \\ 3+12 & 6+16 \end{bmatrix} = \begin{bmatrix} a & 2a \\ 3a & 4a \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix}$$

$$\begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} = \begin{bmatrix} a+b & 2a \\ 3a & 4a+b \end{bmatrix}$$

Thus $a + b = 7$ and $2a = 10$

$\therefore \quad a = 5$ and $b = 2$

Checking for consistency:

$$3a = 3\,(5) = 15 \quad \checkmark$$

$$4a + b = 4\,(5) + (2) = 22 \quad \checkmark$$

Homework page 328 #12, 13, and 14 [show all steps!]