Geometric Sequences

Our textbook defines a geometric sequence as follows:
A sequence is geometric if each term can be obtained from the previous one by multiplying by the same non-zero constant.

In other words, the sequence $\{u_n\}$ is geometric if
$u_{n+1} = r\,u_n$, $r \neq 0$. [Note: $a = u_n$ in some textbooks]
r is called the common ratio and a or u_1 is once again, the first term.

Consider the following sequences:
1, 5, 25, 125, 625, … Is this a geometric sequence?
If it is a geometric sequence, then what is the value of r?

5, 1, $\frac{1}{5}$, $\frac{1}{25}$, $\frac{1}{125}$,..

6, 9, 13.5, 20.25, 30.375, 45.5625, …

If $u_1 = a$, then

$u_2 = r u_1 = a r^1$

$u_3 = r u_2 = a r^2$

$u_4 = r u_3 = a r^3$

$u_5 = r u_4 = a r^4$

Therefore, our general term, $u_n = r u_{n-1} = a r^{n-1}$

Our textbook's definition: [page 44]

Algebraic definition:

$\{u_n\}$ is geometric $\Leftrightarrow$ $\dfrac{u_{n+1}}{u_n} = r$ for all positive integers n

where r is a **constant** (the **common ratio**).

Why are these sequences called "geometric"?
Let $a, b, and\ c$ be any three consecutive terms of a given geometric sequence. [Think about right triangles now.]

$\dfrac{b}{a} = \dfrac{c}{b}$ Now let's cross multiply the proportion

$b^2 = ac$ Which means that

$b = \pm\sqrt{ac}$ So, $\sqrt{ac}$ is the geometric mean

Let's see if this is true with our original geometric sequence 1, 5, 25. $a = 1$, $b = 5$, $c = 25$. Does $\dfrac{b}{a} = \dfrac{c}{b}$?

$\frac{5}{1} = \frac{25}{5}$ This is true statement. Yay!

Notice that the value of r is the value of the geometric mean.

Let's consider Example 6 on page in our textbook:

Example 6

For the sequence $8, 4, 2, 1, \frac{1}{2}, \ldots\ldots$

a Show that the sequence is geometric.

b Find the general term u_n.

c Hence, find the 12th term as a fraction.

a $\frac{4}{8} = \frac{1}{2}$, $\frac{2}{4} = \frac{1}{2}$, $\frac{1}{2} = \frac{1}{2}$, $\frac{\frac{1}{2}}{1} = \frac{1}{2}$

So, assuming the pattern continues, consecutive terms have a common ratio of $\frac{1}{2}$.

$\therefore$ the sequence is geometric with $u_1 = 8$ and $r = \frac{1}{2}$.

b $u_n = u_1 r^{n-1}$ $\quad \therefore \quad u_n = 8\left(\frac{1}{2}\right)^{n-1}$ *or* $u_n = 2^3 \times (2^{-1})^{n-1}$

$$= 2^3 \times 2^{-n+1}$$
$$= 2^{3+(-n+1)}$$
$$= 2^{4-n}$$

c $u_{12} = 8 \times (\frac{1}{2})^{11}$

$$= \frac{8}{2^{11}}$$
$$= \frac{1}{256}$$

(See chapter 3 for exponent simplification.)

Show that the sequence 8, $-$**6**, 4.5, -3.375, ... is geometric and hence find the 10$^{\text{th}}$ term, u_{10}, as a decimal.

Let's first show that the sequence is geometric. Remember that you must justify and that simply using your TI is not considered justification.

Let $a = 8,\ b = -6,\ and\ \ c = 4.5$

Does $\frac{b}{a} = \frac{c}{b}$

$$\frac{-6}{8} = \frac{4.5}{-6}$$ This is true. Our geometric mean = - 0.75

Hence, $r = -0.75$ and $u_1 = a = 8$

Let's use our handy formula to find the general term!

$$u_n = u_1 r^{n-1}$$
$$u_n = (8)(-0.75)^{n-1}$$

Now we need to find the tenth term, u_{10}
Once again, our handy formula will help.

$$u_{10} = (8)(-0.75)^{10-1}$$
$$u_{10} = -.6006774902$$

If you want to, you can check our solutions with a TI.

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seq(8*-.75^(X-1)
,X,1,10,1)
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Now let's look at Example 7 on page 46.

Example 7

$k-1$, $2k$ and $21-k$ are consecutive terms of a geometric sequence. Find k.

Since the terms are geometric, $\dfrac{2k}{k-1} = \dfrac{21-k}{2k}$ {equating r's}

$$\therefore \quad 4k^2 = (21-k)(k-1)$$
$$\therefore \quad 4k^2 = 21k - 21 - k^2 + k$$
$$\therefore \quad 5k^2 - 22k + 21 = 0$$
$$\therefore \quad (5k-7)(k-3) = 0$$
$$\therefore \quad k = \tfrac{7}{5} \text{ or } 3$$

Check: If $k = \frac{7}{5}$ terms are: $\frac{2}{5}, \frac{14}{5}, \frac{98}{5}$. ✓ $\{r = 7\}$

If $k = 3$ terms are: 2, 6, 18. ✓ $\{r = 3\}$

You try page 46 #6c

Find k given that the following are consecutive terms of a geometric sequence: $k,\ k+8,\ 9k$

Find the following:
(a) the value of the 6th term of the geometric sequence whose 3rd term is 15 and whose 4th term is 12.
(b) the value of the first term

We know that $u_3 = 15$ and $u_4 = 12$

We also know that $u_3 = ar^2$ and $u_4 = ar^3$

Then if we do an neat algebraic trick, we can find r.

$$r = \frac{u_{n+1}}{u_n} \quad \text{so, } r = \frac{u_4}{u_3}$$

$$r = \frac{12}{15}$$

$$r = \frac{4}{5}$$

Now we can find the first term, $a \text{ or } u_1$. Since $u_3 = ar^2$,

then $$15 = a\left(\frac{4}{5}\right)^2$$

Hence, $a = u_1 = \frac{375}{16}$ and our general term is

$$u_n = \left(\tfrac{375}{16}\right)\left(\tfrac{4}{5}\right)^{n-1}$$

We can easily find the 6th term now!

$$u_6 = \left(\tfrac{375}{16}\right)\left(\tfrac{4}{5}\right)^{6-1}$$

$$u_6 = \frac{192}{25}$$

Let's consider Example 8 on pages 46, 47

Example 8

A geometric sequence has $u_2 = -6$ and $u_5 = 162$. Find its general term.

$u_2 = u_1 r = -6$ .... (1) {using $u_n = u_1 r^{n-1}$ with $n = 2$}

and $u_5 = u_1 r^4 = 162$ .... (2)

So, $\frac{u_1 r^4}{u_1 r} = \frac{162}{-6}$ {(2) ÷ (1)}

$\therefore \quad r^3 = -27$

$\therefore \quad r = \sqrt[3]{-27}$

$\therefore \quad r = -3$

and so in (1) $u_1(-3) = -6$ $= u_2$

$\therefore \quad u_1 = 2$ $= a$

Thus $u_n = 2 \times (-3)^{n-1}$.

Note: $(-3)^{n-1} \neq -3^{n-1}$

as we do not know the value of n.

If n is odd, then $(-3)^{n-1} = 3^{n-1}$

If n is even, then $(-3)^{n-1} = -3^{n-1}$

TRICKY

A real-life application of geometric sequences – Compound Interest [Remember, with compound interest, you earn interest on your previous interest.]

You invest \$1000 in the bank. You plan on leaving the money in the bank for 4 years [the time you will be college] You are paid 5% compound interest.

At time $t = 0$ you have \$1000.

At time $t = 1$ [after one year] you will have
$(\$1000)(1.05) = \$1050$

At time $t = 2$ [after two years] you will have
$(\$1050)(1.05) = \$1102.50$ which can be obtained by
$(\$1000)(1.05)(1.05)$ or $(\$1000)(1.05)^2$

At time $t = 3$ [after 3 years] you will have
$(\$1102.50)(1.05) = \$1157.63$ which can be obtained by
$(\$1000)(1.05)^3$

At time $t = 4$ [after four years] you will have
$(\$1157.63)(1.05) = \$1215.51$ which can be obtained by
$(\$1000)(1.05)^4$

This is just a geometric sequence with the first term being our initial investment and our ratio being our "growth multipler"

Our general term will be $u_n = (1000)(1.05)^{n-1}$ for $n = 1, 2, 3, \ldots$

Note: This looks different than the $A = Pe^{rt}$ formula you learned for compound interest in Pre-Calculus.

Let's look at some examples from our textbook: [page 48]

Example 10

PER ANNUM

\$5000 is invested for 4 years at 7% p.a. compound interest.
What will it amount to at the end of this period?

$u_5 = u_1 \times r^4$ is the amount after 4 years

$= 5000 \times (1.07)^4$ {for a 7% increase 100% becomes 107%}

$\doteqdot 6553.98$ {5000 [×] 1.07 [^] 4 [=] }

So, it amounts to \$6553.98.

In this case, $u_1 = 5000$ *and* $r = 1.07$ so our general term is

$$u_n = (5000)(1.07)^{n-1} \text{ for } n = 1, 2, 3, \ldots$$

COMPARE TO
$A = Pe^{rt}$
$A = 5000e^{(.07)(4)}$
COMPOUND CONTINUOUS

The amount of interest earned is
\$6553.98 - \$5000=\$1553.98

Example 11

How much should I invest now if I want the maturing value to be \$10 000 in 4 years' time, if I am able to invest at 8.5% p.a. compounded annually?

$u_1 = ?$, $u_5 = 10\,000$, $r = 1.085$

$t=5$

$u_5 = u_1 \times r^4$ {using $u_{n+1} = u_1 \times r^n$}

$\therefore \quad 10\,000 = u_1 \times (1.085)^4$

$\therefore \quad u_1 = \dfrac{10\,000}{(1.085)^4}$

$\doteqdot$ means $\approx$

$\therefore \quad u_1 \doteqdot 7215.74$ {10 000 [÷] 1.085 [^] 4 [=]}

$u_n = u_1 r^{n-1}$

So, you should invest \$7215.74 now.

Let's look at page 49 #15.
How much money must be invested now if you require \$20000 for a holiday in 4 years' time and the money can be invested at a fixed rate of 7.5% p.a. compounded annually? [I want to go on "holiday" with this person.]

We know that $u_5 = \$20000$ and $r = 1.075$

$$\$20000 = a\,r^4$$

$$\$20000 = a\,(1.075)^4$$

So

$$\frac{20000}{(1.075)^4} = a = u_1$$

Hence, \$14976.01 is the approximate amount of money that you will need to invest in order to go on a fabulous \$20000 holiday in four years.

If time, then do the "bunny problem" on page 50.

Example 12

The initial population of rabbits on a farm was 50.
The population increased by 7% each week.

a How many rabbits were present after:

 i 15 weeks **ii** 30 weeks?

b How long would it take for the population to reach 500?

We notice that $u_1 = 50$ and $r = 1.07$

$u_2 = 50 \times 1.07 =$ the population after 1 week

a **i** $u_{n+1} = u_1 \times r^n$

$\therefore \quad u_{16} = 50 \times (1.07)^{15}$

$\doteqdot 137.95....$

i.e., 138 rabbits

ii and

$u_{31} = 50 \times (1.07)^{30}$

$\doteqdot 380.61....$

i.e., 381 rabbits

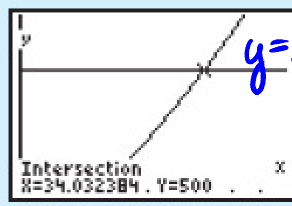
b $u_{n+1} = u_1 \times (1.07)^n$ after n weeks

So, we need to find when $50 \times (1.07)^n = 500$.

Trial and error on your calculator gives $n \doteqdot 34$ weeks

or using the **Equation Solver** gives $n \doteqdot 34.03$

or by finding the **point of intersection** of $Y_1 = 50 \times 1.07$^X and $Y_2 = 500$ on a graphics calculator, the solution is $\doteqdot 34.03$ weeks.



$u_n = u_1 r^{n-1}$

$n = 1$

$u_{n+1} = u_1 r^n$

Some things to note:
You can't have a "partial" bunny.
This would probably be a non-calculator problem but you might be required to solve algebraically [using logs].

Homework: page 46 #1, 3, 5, page 47 #7a, 7b, 7c, 8a, page 49 #9, 11, 17, page 50 #19

1 For the geometric sequence with first two terms given, find b and c:

a $2, 6, b, c, ...$ **b** $10, 5, b, c, ...$ **c** $12, -6, b, c, ...$

3 **a** Show that the sequence $12, -6, 3, -1.5, ...$ is geometric.

b Find u_n and hence find the 13th term (as a fraction).

5 Show that the sequence $8, 4\sqrt{2}, 4, 2\sqrt{2}, ...$ is geometric and hence find, in simplest form, the general term u_n.

7 Find the general term u_n, of the geometric sequence which has:

a $u_4 = 24$ and $u_7 = 192$ **b** $u_3 = 8$ and $u_6 = -1$

c $u_7 = 24$ and $u_{15} = 384$ **d** $u_3 = 5$ and $u_7 = \frac{5}{4}$

8 **a** Find the first term of the sequence $2, 6, 18, 54, ...$ which exceeds $10\,000$.

9 **a** What will an investment of \$3000 at 10% p.a. compound interest amount to afte 3 years?

b What part of this is interest?

11 **a** What will an investment of 30 000 yen at 10% p.a. compound interest amount to after 4 years?

b What part of this is interest?

17 How much should I invest now if I want a maturing amount of 25 000 Euro in 3 years' time and the money can be invested at a fixed rate of 8% p.a. compounded quarterly?

19 A nest of ants initially consists of 500 ants. The population is increasing by 12% each week.

a How many ants will there be after

i 10 weeks **ii** 20 weeks?

b Use technology to find how many weeks it will take for the ant population to reach 2000.