Arithmetic Sequences

Definition:
A sequence, $\{u_n\}$, which is defined in the form
$u_{n+1} = u_n + d$ where d is a constant, is called an ***arithmetic sequence***. Sometimes the first term, u_1 is denoted as a.

$d = 1$

In general, [Note: $n = \{1, 2, 3, 4, \ldots\}$ or $n \in Z^+$
$u_1 = a$ and
$u_2 = u_1 + d = a + d$
$u_3 = u_2 + d$
$u_4 = u_3 + d$ and so on

2, 4, 6, 8, ...
$u_1 = 2$
$d = 2$

The general term, u_n, will equal: $u_n = u_{n-1} + d$

Our textbook's definition: [from Haese, Harris, et al]

ALGEBRAIC DEFINITION

$\{u_n\}$ is **arithmetic** $\Leftrightarrow$ $u_{n+1} - u_n = d$ for all positive integers n where d is a constant (the **common difference**).

Note:
- $\Leftrightarrow$ is read as 'if and only if'
- If $\{u_n\}$ is arithmetic then $u_{n+1} - u_n$ is a constant *and* if $u_{n+1} - u_n$ is a constant then $\{u_n\}$ is arithmetic.

Consider the sequence: 7, 11, 15, 19, 23, …

What is u_1 [or a] 7

What is the common difference, d? 4

What is the general term, u_n?

$u_{n+1} = u_n + d$

$u_{n+1} = u_n + 4$

$u_{n+1} - u_n = d$

$23 - 19 = 4$

Is $u_2 = u_1 + d$?

$11 = 7 + 4$

Is $u_3 = u_2 + d$

$15 = 11 + 4$

$15 = 7 + 2(4)$

And now for the handiest formula: [once again, from our textbook by Haese, Harris, et al]

THE GENERAL TERM FORMULA

Suppose the first term of an arithmetic sequence is u_1 and the common difference is d.

Then $u_2 = u_1 + d$ $\therefore$ $u_3 = u_1 + 2d$ $\therefore$ $u_4 = u_1 + 3d$ etc.

then, $u_n = u_1 + \underbrace{(n-1)}d$

The coefficient of d is one less than the subscript.

So, for an **arithmetic sequence** with **first term** u_1 and **common difference** d the **general term** (or nth **term**) is $u_n = u_1 + (n-1)d$.

Find u_{100} $n = 100$

$u_{100} = 7 + (100-1)(4)$

$= 7 + (99)(4)$

$= 403$

Let's see how useful this formula is!

Find the 10^{th} term of the sequence defined by $u_n = 4n+3$ where $n \in Z^+$.

Since we need u_1, we should find it first.

$$u_1 = 4(1)+3$$

$$u_1 = 7$$

Our sequence looks like: 7, 11, 15, …

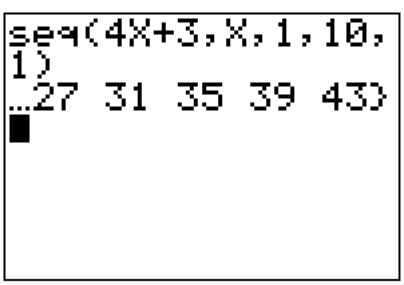
Now we could easily list the first ten terms but let's use our formula instead.

$$u_n = u_1 + (n-1)d$$

In this case: $u_1 = 7$, $n = 10$, $d = 4$

So, $$u_{10} = 7 + (10-1)(4)$$ $$u_{10} = 43$$

We can easily verify this with our TI:



2nd LIST, OPS

Now let's consider a slightly harder problem:
Find the 10th term of an arithmetic sequence whose first term, a, is 6, and whose common difference, d is -3.

u_1

We have everything we need!

$u_n = u_1 + (n-1)d$ [Remember, $a = u_1$]

$u_{10} = 6 + (10-1)(-3)$

$u_{10} = -21$

What would our general term be? in terms of n

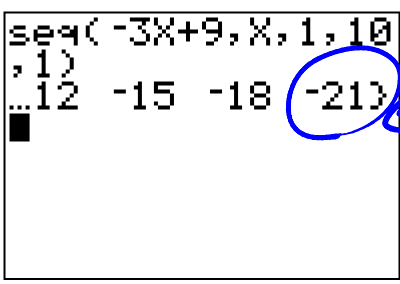
$u_n = u_1 + (n-1)d$

$u_n = 6 + (n-1)(-3)$

$u_n = 6 - 3n + 3$ $n \in \mathbb{Z}^+$

$u_n = -3n + 9$

We can verify our answer with our TI.



u_{10}

Let's consider Example 2 from our textbook, page 42.

Example 2

Consider the sequence 2, 9, 16, 23, 30, ...

a Show that the sequence is arithmetic.

b Find the formula for the general term u_n.

c Find the 100th term of the sequence.

d Is **i** 828 **ii** 2341 a member of the sequence?

a
$9 - 2 = 7$
$16 - 9 = 7$
$23 - 16 = 7$
$30 - 23 = 7$

So, assuming that the pattern continues, consecutive terms differ by 7

$\therefore$ the sequence is arithmetic with $u_1 = 2$, $d = 7$.

IN TERMS OF n

b $u_n = u_1 + (n-1)d$ $\therefore$ $u_n = 2 + 7(n-1)$ i.e., $u_n = 7n - 5$

$n \in \mathbb{Z}^+$

c If $n = 100$, $u_{100} = 7(100) - 5 = 695$.

d **i** Let $u_n = 828$
$\therefore$ $7n - 5 = 828$
$\therefore$ $7n = 833$
$\therefore$ $n = 119$

yay! $n \in \mathbb{Z}^+$

$\therefore$ 828 is a term of the sequence. In fact it is the 119th term.

ii Let $u_n = 2341$
$\therefore$ $7n - 5 = 2341$
$\therefore$ $7n = 2346$
$\therefore$ $n = 335\frac{1}{7}$

which is not possible as n is an integer. $\therefore$ 2341 cannot be a term.

Please note: Simply using your TI is not sufficient for justification. Don't worry, the formula for the general term is contained in the official IB formula packet.

Let's try one! [from page 42]
Consider the sequence 6, 17, 28, 39, 50, …
Show that the sequence is arithmetic.

$17 - 6 = 11$
$28 - 17 = 11$
$39 - 28 = 11$
$50 - 39 = 11$

$u_1 = a = 6$
$d = 11$

Find the formula for the general term.

$$u_n = u_1 + (n-1)d$$
$$u_n = 6 + (n-1)(11)$$
$$u_n = 11n - 5$$

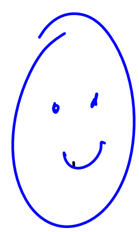
n^{th} is the general TERM

Find the 50^{th} term [or u_{50}].

$$u_n = 11n - 5$$
$$u_{50} = 11(50) - 5$$
$$u_{50} = 545$$

Is 325 a member?

$$325 = 11n - 5$$
$$330 = 11n$$
$$30 = n$$



yay!

Is 761 a member?

$$761 = 11n - 5$$
$$766 = 11n$$
$$69.64 \approx n$$

NO!

$$n \in \mathbb{Z}^+$$

Now for something slightly different!

Find k, given the consecutive arithmetic terms:
$k+1,\ 2k+1,\ 13$
$u_1 \quad u_2 \quad u_3$

Since this is an arithmetic sequence, then we can equate the common differences. Remember, $d = u_{n+1} - u_n$. So we can subtract pairs of consecutive terms and equate them to each other. In this case,
$(2k+1)-(k+1)=13-(2k+1)$
$u_2 - u_1 = u_3 - u_2$

This leaves us with:
$k=12-2k$ which can be easily solved!
$3k=12$

$k=4$

$u_1 = k+1 = 4+1 = 5 > 4$
$u_2 = 2k+1 = 8+1 = 9 > 4$
$u_3 = 13$

We should check our solution. Let $k=4$, then our sequence is 5, 9, 13 which is an arithmetic sequence whose common difference, d, is 4.

$u_n = 5 + (n-1)(4)$
$u_n = 4n+1$

What if we were given the following information?
$u_7 = 1$ and $u_{15} = -39$. Find u_n

We know that $u_n = u_1 + (n-1)d$

That means, that $\begin{aligned} u_7 &= u_1 + 6d \\ u_{15} &= u_1 + 14d \end{aligned}$

$1 = u_1 + 6d$
$-39 = u_1 + 14d$

We can write two equations with two unknowns and solve the system of equations!

$$\begin{cases} 1 = u_1 + 6d \\ -39 = u_1 + 14d \end{cases}$$

Use any method to solve, and we should get that $\boxed{u_1 = 31}$ and $\boxed{d = -5.}$ So $u_n = 31 + (n-1)(-5)$ or $u_n = 36 - 5n$

31, 26, 21, 16, 11, 6, 1, -4, -9, ...

Homework: page 42 #2, 3, 4, and page 43 #6a and 6b, and page 44 #7a, 7b and #9 You might want to read the section first. WRITE DOWN FORMULAS

2 Consider the sequence 87, 83, 79, 75, ...

- **a** Show that the sequence is arithmetic.
- **b** Find the formula for the general term.
- **c** Find the 40th term.
- **d** Is -143 a member?

3 A sequence is defined by $u_n = 3n - 2$.

- **a** Prove that the sequence is arithmetic. (**Hint:** Find $u_{n+1} - u_n$.)
- **b** Find u_1 and d.
- **c** Find the 57th term.
- **d** What is the least term of the sequence which is greater than 450?

4 A sequence is defined by $u_n = \dfrac{71 - 7n}{2}$.

- **a** Prove that the sequence is arithmetic.
- **b** Find u_1 and d.
- **c** Find u_{75}.
- **d** For what values of n are the terms of the sequence less than -200?

6 Find the general term u_n for an arithmetic sequence given that:

- **a** $u_7 = 41$ and $u_{13} = 77$
- **b** $u_5 = -2$ and $u_{12} = -12\frac{1}{2}$

7 **a** Insert three numbers between 5 and 10 so that all five numbers are in arithmetic sequence.

b Insert six numbers between -1 and 32 so that all eight numbers are in arithmetic sequence.

9 An arithmetic sequence starts 23, 36, 49, 62, ... What is the first term of the sequence to exceed 100 000?