Inverse Functions

If $y = x - 5$, then what is y^{-1}?

$x = y - 5$

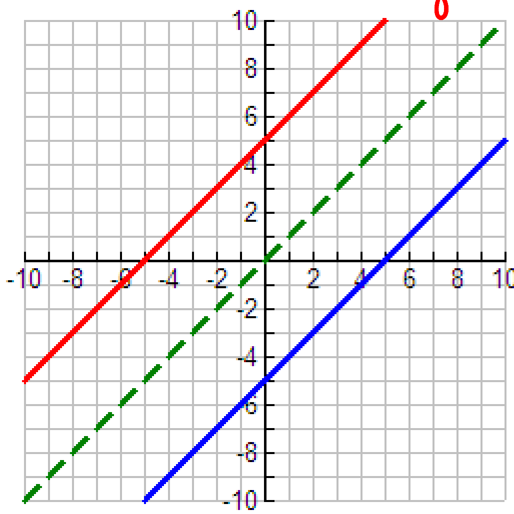
$x + 5 = y$

$f^{-1}(x) = x + 5$

$y^{-1} = x + 5$ $(-5, 0)$

$y = x$

$y = x - 5$ $(0, -5)$



$f^{-1}(x)$ is a reflection across the line $y = x$.

If the graph of f contains the point (a, b), then the graph of f^{-1} contains the point (b, a)

$f(f^{-1}(x)) = x$

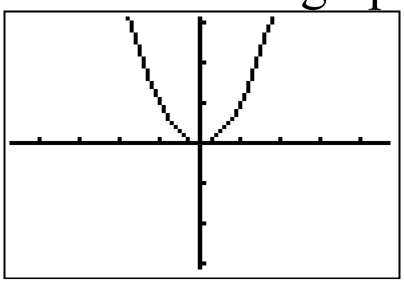
or $f(g(x)) = x$ implies $g = f^{-1}$

x	$f(x) = x - 5$
0	- 5
5	0

x	$f^{-1}(x) = x + 5$
-5	0
0	5

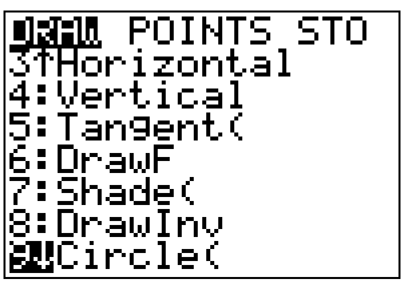
Every function has an inverse [but it may not be an inverse function]

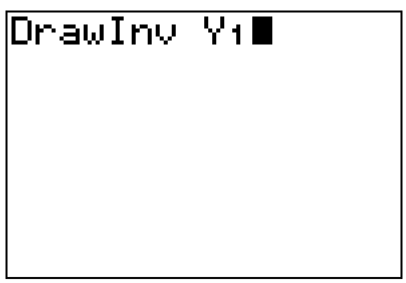
Your TI will graph any inverse for you.



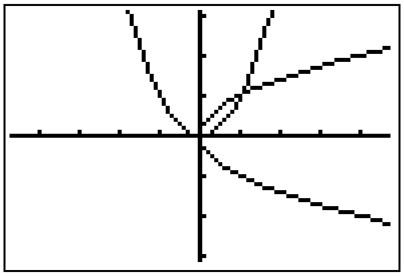
Let $y_1 = x^2$

Go to the DRAW Menu and choose option 8 “DrawInv”





We’ll get this result:



Notice that the inverse is not a function

If $f(x) = x^2$, then $f^{-1}(x) = \sqrt{x}$ $\quad x \geq 0$

$f'(x) = 2x$ and $\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{2\sqrt{x}}$

$y = x^2$
$x = y^2$
$\sqrt{x} = y$

x	$f(x)$ $y=x^2$	$f'(x)$ $y'=2x$	☺	x	$f^{-1}(x)$ $y^{-1}=\sqrt{x}$	$\frac{d}{dx}\left(f^{-1}(x)\right)$
1	1	2	☼	1	1	$\frac{1}{2}$
2	4	4	♥	4	2	$\frac{1}{4}$
3	9	6	♫	9	3	$\frac{1}{6}$
4	16	8	⁂	16	4	$\frac{1}{8}$

Compare the derivative columns. Hmm!

Let $f(x) = x^3$, then $f^{-1}(x) = \sqrt[3]{x}$

$f'(x) = 3x^2$ and $\dfrac{d}{dx}\left(f^{-1}(x)\right) = \dfrac{1}{3}x^{-\frac{2}{3}}$

$= \dfrac{1}{3\sqrt[3]{x^2}}$

Let's make a chart to compare again.

x	$f(x)$ $y=x^3$	$f'(x)$ $=3x^2$	☯	x	$f^{-1}(x)$	$\dfrac{d}{dx}\left(f^{-1}(x)\right)$
1	1	3	✵	1	1	$\frac{1}{3}$
2	8	12	ω	8	2	$\frac{1}{12}$
3	27	27	↯	27	3	$\frac{1}{27}$

$f(2) = 8$ $\qquad g = f^{-1}$ $\qquad g(8) = 2$

$f'(2) = 12$ $\qquad g'(8) = \dfrac{1}{f'(2)}$

Theorem

Let f be a function that is differentiable on an interval, I. If f has an inverse *function* g, then g is differentiable at any x for which $f'(g(x)) \neq 0$ AND

$$g'(x) = \frac{1}{f'(g(x))}$$

If we are lucky, then we can figure out what the inverse function is, but most of the time, we will not have this option.

The theorem in action:
$f(x) = x^3, \ f'(x) = 3x^2$

The point $(5,125)$ is on the graph of $f(x)$ and $f'(5) = 75$. From the theorem, we know that the point $(125,5)$ is on the graph of $f^{-1}(x)$ and if we let $g(x) = f^{-1}(x)$, then we also know that $g'(125) = \frac{1}{75}$ or $g'(125) = \frac{1}{f'(5)}$
Notice that we did not need to find the inverse function.

If $f(3) = 7, \ f'(3) = -10$, and $g = f^{-1}$, then what do we know about $f^{-1}(x) = g(x)$ and $g'(x)$?

$g(7) = 3$

$g'(7) = \frac{1}{f'(3)} = -\frac{1}{10}$

If $f(20) = 12, \quad f'(20) = 2$ and $g = f^{-1}$, then what do we know?

$g(12) = 20$

$g'(12) = \frac{1}{f'(20)}$

$g'(12) = \frac{1}{2}$

If $f(3)=0,\ f'(3)=-6$, then

$g(0) = 3$

$g'(0) = \dfrac{1}{f'(3)}$

$g'(0) = \dfrac{1}{-6}$

If $f(12)=-1,\ f'(12)=6$, then

$g(-1) = 12$

$g'(-1) = \dfrac{1}{f'(12)}$

$g'(-1) = \dfrac{1}{6}$

Slightly different:

Let $g(x) = f^{-1}(x)$ and find the appropriate value of $g'(x)$

$f(x) = x + \cos x$ Find $g'(1)$

need $1 = x + \cos x$

$0 = x$

$(0,1)$ on $f(x)$

$f'(x) = 1 - \sin x$

$f'(0) = 1$

$g'(1) = \dfrac{1}{f'(0)}$

$g'(1) = 1$

$f(x)=\sqrt{x^2+6x}$ where $x \geq 0$ Find $g'(4)$

$4=\sqrt{x^2+6x}$ $(2,4)$ on $f(x)$

$g'(4)=\dfrac{1}{f'(2)}$ $f'(x)=\dfrac{x+3}{\sqrt{x^2+6x}}$ $f'(2)=\dfrac{5}{4}$

$g'(4)=\dfrac{4}{5}$

My Inverse Function Worksheet ____

Let $g(x)=f^{-1}(x)$ and use the given information to find the appropriate value of $g'(x)$

1. Given: $f(x)=x^4,\ f(2)=16$

2. $f(x)=3-4x,\ f(1)=-1$

3. $f(x)=\sqrt{x-4},\ f(5)=1$

4. $f(x)=\sin x,\ f\left(\dfrac{\pi}{3}\right)=\dfrac{\sqrt{3}}{2}$

5. $f(x) = x^2,\ f(6) = 36$

6. $f(x) = \ln\left(x^2 + 1\right),\ f(2) = \ln 5$

7. Let $g = f^{-1}$ and $f(x) = 7x^2 - 5x + 3$. Find $g'(21)$ $[(x > 0)]$

8. Let $g = f^{-1}$ and $f(x) = x^2 + 3x + 1$. Find $g'(29)$ $[x > 0]$

9. Let $g = f^{-1}$ and $f(x) = x^3 + 2x^2 - 10$. Find $g'(6)$ $[x > 0]$

10. Let $g = f^{-1}$ and $f(x) = 6x^2 + 4x - 2$. Find $g'(8)$ $[x > 0]$

11. Let $g = f^{-1}$ and $f(x) = 5x^3 - 10x^2$. Find $g'(45)$ $[x > 0]$

12. Let $g = f^{-1}$ and $f(x) = x^4 - 5x$. Find $g'(6)$ $[x > 0]$