Infinite Limits

Vertical Asymptote [Calculus definition]

If $\lim_{x \to c} f(x) = \infty$ or $\lim_{x \to c} f(x) = -\infty$, then $x = c$ is a vertical asymptote of $f(x)$. Where $c \in$ Reals.

Note: This works if $x \to c$ from the left or the right or both.

Another note: ∞ is NOT a number. If $\lim_{x \to c} f(x) = \infty$, then that means that the function grows without bound as x approaches c

How can you "spot" a vertical asymptote?

If $f(c)$ is the undefined form $\dfrac{\textit{nonzero number}}{\textit{zero}}$, then $x = c$ is a vertical asymptote.

Let's consider some functions.

$f(x) = \tan x$ on $[0, \pi]$

$$\lim_{x \to \frac{\pi}{2}^-} \tan x = \infty \quad \text{AND} \quad \lim_{x \to \frac{\pi}{2}^+} \tan x = -\infty$$

Hence, $x = \dfrac{\pi}{2}$ is a vertical asymptote of $f(x) = \tan x$ on $[0, \pi]$

Let $g(x) = \dfrac{1}{x^3}$ Domain: $(-\infty, 0) \cup (0, \infty)$

$$\lim_{x \to 0^-} g(x) = -\infty \quad \text{AND} \quad \lim_{x \to 0^+} g(x) = \infty$$

$g(-.01) < 0$ $g(.01) > 0$

$x = 0$ is A VERTICAL Asymptote

Slightly harder – let $h(x) = \dfrac{-4x}{x^2 - 4}$

Domain: $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

$$\lim_{x \to -2^-} h(x) = \infty \quad \text{AND} \quad \lim_{x \to -2^+} h(x) = -\infty$$

$$\lim_{x \to 2^-} h(x) = \infty \quad \text{AND} \quad \lim_{x \to 2^+} h(x) = -\infty$$

$h(-2.1) > 0$ $h(-1.9) < 0$

$h(1.9) > 0$ $h(2.1) < 0$

$h(x)$ has V. A. at $x = \pm 2$

We need to be careful with rational functions.

Let $f(x) = \dfrac{x^2 - 2x}{x^2 - 4}$

Domain: $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

Same domain as the previous problem but we need to actually find limits to determine any vertical asymptotes.

$$\lim_{x \to 2} \frac{x^2 - 2x}{x^2 - 4}$$

$$= \lim_{x \to 2} \frac{x\cancel{(x-2)}}{\cancel{(x-2)}(x+2)}$$

$$= \lim_{x \to 2} \frac{x}{x+2}$$

$$= \frac{1}{2}$$

$f(2)$ undefined

$\lim_{x \to 2} f(x) = \frac{1}{2}$

Remov DISC at $\left(2, \frac{1}{2}\right)$

$\lim_{x \to -2^-} f(x) = \infty$ AND $\lim_{x \to -2^+} f(x) = -\infty$

$f(-2.1)$ $\qquad$ $f(-1.9)$

f has a V.A. at $x = -2$

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An "AP" Example:

$\lim_{x \to 0} \frac{1}{x^2}$ is

(A) 0 (B) 1 (C) π (D) e (E) does not exist

Let do page 88 #1 – 4 together:

(1) $\lim_{x \to -2^-} f(x) = \infty$

$\lim_{x \to -2^+} f(x) = \infty$

(2) $\lim_{x \to -2^-} f(x) = -\infty$

$\lim_{x \to -2^+} f(x) = \infty$

(3) $\lim_{x \to -2^-} f(x) = \infty$

$\lim_{x \to -2^+} f(x) = -\infty$

(4) $\lim_{x \to -2^-} f(x) = -\infty$

$\lim_{x \to -2^+} f(x) = \infty$

Now let's hunt for infinite discontinuities or vertical asymptotes.

VERTICAL ASYMPTOTE

We don't need a gun, just brains and limits, to find vertical asymptotes!

$$f(x) = \frac{x^2-1}{x+1}$$

$$\lim_{x \to -1} \frac{\cancel{(x+1)}(x-1)}{\cancel{x+1}} = -2$$

R.D. at $(-1,-2)$

$$g(x) = \frac{x-1}{x-5}$$

$$\lim_{x \to 5^-} g(x) = -\infty$$

$$\lim_{x \to 5^+} g(x) = \infty$$

$g(4.9) < 0$

$g(5.1) > 0$

$$h(x) = \frac{(x+3)(x-2)}{(x+1)(x-2)}$$

R.D. at $(2, \frac{5}{3})$

$$\lim_{x \to 2} \frac{(x+3)\cancel{(x-2)}}{(x+1)\cancel{(x-2)}}$$

$$\lim_{x \to -1^-} h(x) = -\infty$$

$h(-1.1) < 0$

$$= \lim_{x \to 2} \frac{x+3}{x+1}$$

$$= \frac{5}{3}$$

$$\lim_{x \to -1^+} h(x) = \infty$$

$h(-.9)$

Homework: Read 1.5 and do page 88 # 33, 35, 37, 39, 41