Making Functions Continuous

Consider: $f(x)=\dfrac{\sin x}{x}$

Domain: $(-\infty, 0) \cup (0, \infty)$

Let's rewrite as a piecewise function

$$g(x)=\begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ a, & x = 0 \end{cases}$$

$a = 1$

What should the value of a be such that $g(x)$ is continuous for all real numbers? Well, we only know one calculus "trick" so we should probably use it.

To be continuous at $x = 0$, what must be true?

$$f(0) = \lim_{x \to 0} \frac{\sin x}{x}$$

Let's try some ! $\lim_{x \to 0} \frac{\sin x}{x} = 1$

$$f(x)=\begin{cases}\dfrac{x^2-9}{x-3}, & x\neq 3\\ b, & x=3\end{cases}$$

$b=6$

$$\lim_{x\to 3}\frac{x^2-9}{x-3}$$

$$=\lim_{x\to 3}\frac{(x+3)\cancel{(x-3)}}{\cancel{x-3}}$$

need $f(3)$

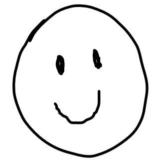
$=\lim_{x\to 3} f(x)$

$$=\lim_{x\to 3}(x+3)$$

$=6$

$$g(x) = \begin{cases} x^2 - 5x + 3, & x \le 2 \\ ax + 1, & x > 2 \end{cases}$$

$-2x+1$



need $g(2) = \lim_{x \to 2} g(x)$

$g(2) = 4 - 10 + 3$

$g(2) = -3$

$\lim_{x \to 2^-} g(x) = -3$

$\lim_{x \to 2^+} g(x) = 2a + 1$

$-3 = 2a + 1$

$-2 = a$

$$h(x) = \begin{cases} \dfrac{k}{x^2}, & x < -2 \\ 9 - x^2, & x \geq -2 \end{cases}$$

$h(-2) = 5$

need $h(-2) = \lim_{x \to -2} h(x)$

$\lim_{x \to -2^-} h(x) = \frac{k}{4}$

$\lim_{x \to -2^+} h(x) = 5$

$\frac{k}{4} = 5$

$k = 20$

Slightly harder:

$$f(x)=\begin{cases} x^2-5x+3, & x<-1 \\ ax+b, & -1\le x\le 4 \\ 11-3x, & x>4 \end{cases}$$

$-2x+7$

need $\lim_{x\to -1} f(x) = f(-1)$

$\lim_{x\to 4} f(x) = f(4)$

$$\lim_{x\to -1^-} f(x) = 9$$

$$\lim_{x\to -1^+} f(x) = -a+b$$

$$\boxed{\begin{aligned} 9 &= -a+b \\ -1 &= 4a+b \end{aligned}}$$

$$\lim_{x\to 4^-} f(x) = 4a+b$$

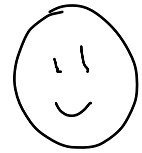
$$\lim_{x\to 4^+} f(x) = -1$$

$a=-2$

$b=7$

$$g(x) = \begin{cases} cx+1, & x \le 3 \\ cx^2 - 1, & x > 3 \end{cases}$$

$$\begin{cases} \frac{1}{3}x+1, & x \le 3 \\ \frac{1}{3}x^2 - 1, & x > 3 \end{cases}$$



$$g(3) = \lim_{x\to 3} g(x)$$

$$\lim_{x\to 3^-} g(x) = 3c+1$$

$$\lim_{x\to 3^+} g(x) = 9c-1$$

$$3c+1 = 9c-1$$

$$2 = 6c$$

$$\frac{1}{3} = c$$

Slightly different:

$$h(x) = \begin{cases} \dfrac{x^2 - a^2}{x-a}, & x \ne a \\ 2, & x = a \end{cases}$$

$$h(a) = 2$$

$$\lim_{x\to a} \frac{x^2-a^2}{x-a}$$

$$= \lim_{x\to a} \frac{(x+a)\cancel{(x-a)}}{\cancel{x-a}}$$

$$= \lim_{x\to a} (x+a)$$

need $h(a) = \lim_{x\to a} h(x)$

$$2a = 2$$

$$a = 1$$

Need more examples? Go to:
http://www.math.ucdavis.edu/~kouba/CalcOneDIRECTORY/continuitydirectory/Continuity.html

Intermediate Value Theorem

This is an existence theorem and will not provide a solution. It just tells us of the existence of a solution.

x-values

y-values

If f is continuous on the closed interval $[a,b]$ and k is a number such that $f(a) < k < f(b)$, then there is at least one number c, such that $a < c < b$, such that $f(c) = k$.

In calculus: Let $f(x) = x^3 - x - 1$. Use the IVT to show that there is at least one root [zero] on $[1,2]$

♪ We must use IVT, so a graph is not sufficient.

$f(1) = 1 - 1 - 1 = -1$

$f(2) = 8 - 2 - 1 = 5$

We know that since $f(x)$ is a polynomial, it is continuous on the interval. [Actually, it is continuous everywhere!] We have found that $f(1) = -1$ and $f(2) = 5$.

Our Calculus-based proof is:

By the IVT, there must exist some c, with $1 < c < 2$, such that $f(1) < f(c) < f(2)$. Which means that there must exist a c, $1 < c < 2$, such that $-1 < f(c) < 5$. Hence, there exists at least one root on [1, 2].

Construct a convincing argument using the IVT to show that if $g(x) = x^3 - 5x^2 + 8x - 9$, then there exists some c such that $g(c) = 27$.

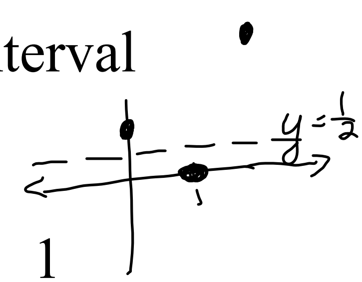
$g(0) = -9$
$g(10) = 571$

By the IVT there is a c, $0 < c < 10$, such that $g(0) < g(c) < g(10)$. Which means there is a c, $0 < c < 10$, such that $-9 < g(c) < 571$
Hence $g(c) = 27$ for $0 < c < 10$.

An example of an IVT question:

x	0	1	2
$f(x)$	1	k	2

The function f is continuous on [0, 2] and has values given in the table above. The equation $f(x) = \frac{1}{2}$ has at least two solutions in the interval [0, 2] if $k =$



(A) 0 (B) $\frac{1}{2}$ (C) 1

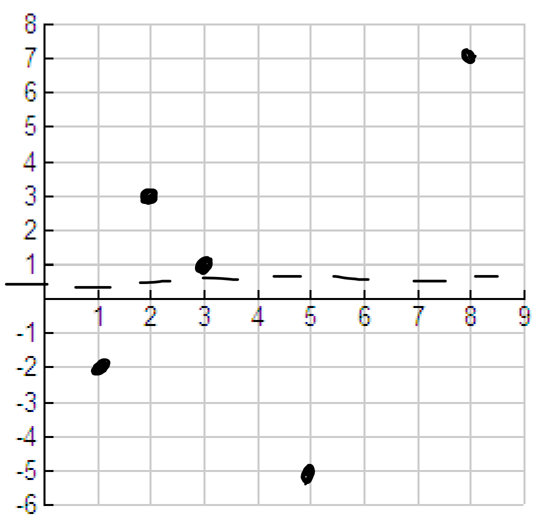
(D) 2 (E) 3

Let f be a continuous function. Selected values of f are given in the table below.

x	1	2	3	5	8
$f(x)$	-2	3	1	-5	7

What is the least number of solutions does the equation $f(x)=\frac{1}{2}$ have on the closed interval $1 \le x \le 8$

[Hint: Draw a picture first]



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Write down this homework assignment but we will first work on our chapter one handout

♪ Remember to always find the one-sided limits for the x – value in question. You may also want to check your solution by graphing it.

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