A cauldron [big pot] of Polyjuice potion is placed on a counter to cool. Let $T(x)$ represent the temperature of the potion at time x, where T is a differentiable function of x. The temperature of the potion at selected times is given in the table below. Notice that the potion is not always cooling because it is a magical potion.

x (hours)	0	2	4	6	8
$T(x)$ [degrees]	200	182	210	170	150

(A) Use data from the table to find the average rate of change of $T(x)$ over the time interval $0 \le x \le 8$. Indicate units.

Average rate of change on [0, 8]

Please do NOT say "$T'(x)$". That is a false statement!

$$ARof\Delta = \frac{T(8) - T(0)}{8 - 0}$$

[Difference quotient must be present in this form]

$$= \frac{150 - 200}{8}$$

$$= \frac{-25}{4} \frac{\deg}{hr}$$

[Needed to include units]

Cannot say that $-6.25 \deg/hr = 6.25 \deg/hr$

(B) Use the data from the table to estimate $T'(4)$ and use it to write the equation of the tangent line to T at $x = 4$.

Need to estimate the slope of a tangent with a slope of a secant. The Δx must be as small as possible. Two values were accepted.

$$T'(4) \approx \frac{T(4) - T(2)}{4 - 2} = 14 \qquad \text{OR} \qquad T'(4) \approx \frac{T(6) - T(4)}{6 - 4} = -20$$

$$y - 210 = 14(x - 4) \quad \text{OR} \quad y - 210 = -20(x - 4)$$

Once again, the difference quotient must be present in this form. I think that some people would benefit from taking better notes in class.

(C) Use the tangent line from part (B) to estimate a value for $T(5)$

$$y - 210 = 14(x - 4) \qquad y - 210 = -20(x - 4)$$

$$\text{Let } x = 5$$

$$y - 210 = 14(5 - 4) \qquad y - 210 = -20(5 - 4)$$

$$T(5) \approx 224^\circ \quad \text{OR} \quad T(5) \approx 190^\circ$$

Most common reasons for deducted points:

Heinous notation

Too big of an interval to estimate the slope of the tangent

Difference quotient missing

Algebraic and computational issues even though you had use of a calculator

Area [definite integrals]

Σ SIGMA

Sigma notation

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_n$$

This is a sum of n- number of terms.

How this works:

$$\sum_{i=1}^{8} i = 1+2+3+4+5+6+7+8 = 36$$

sum of the i's from 1 to 8

$$\sum_{i=2}^{5} i^2 = 2^2 + 3^2 + 4^2 + 5^2 = 54$$

$\int$ is really a stylized S which stands for sum.

So far we have only looked at indefinite integrals but there are definite integrals. For example:

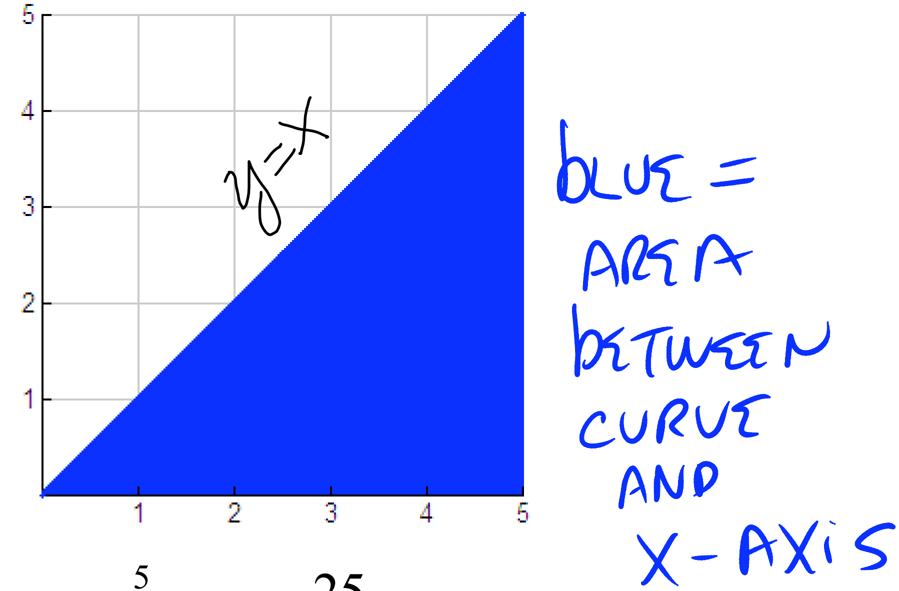
UPPER BOUND

$$\int_0^5 x\,dx =$$

LOWER BOUND

sum of the area between the curve $y = x$ and the $x-$axis
FROM $x=0$ TO $x=5$

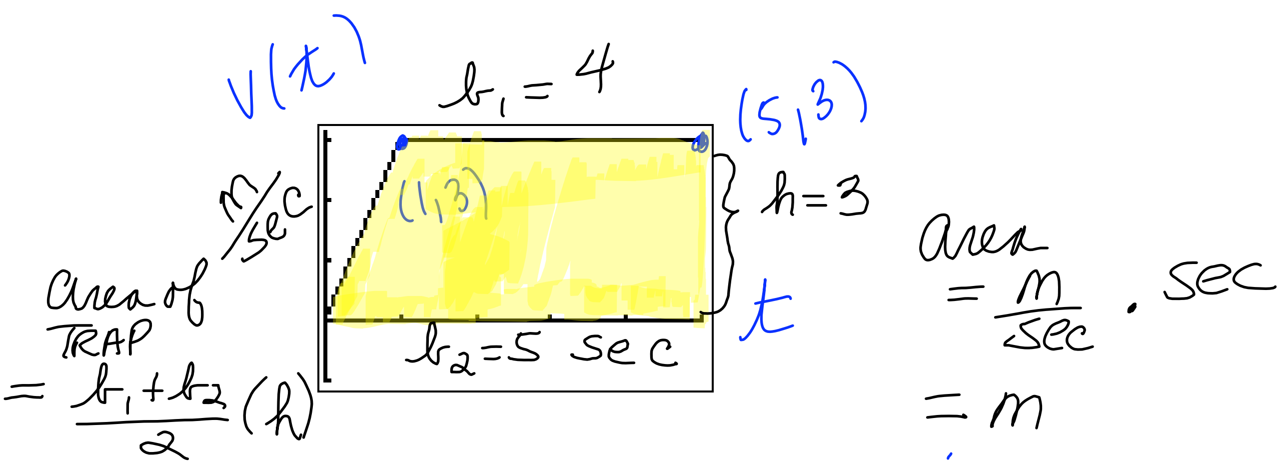
What's that look like? Let's draw a picture.



Area of shaded region = $\int_0^5 x\,dx = \frac{25}{2}$

It's a triangle whose base and height equal 5 units.

Let's consider this graph of $v(t)$ on [0, 5]



To find the total distance traveled on [0, 5] we can simply find $\int_0^5 v(t)\,dt$ [♪ we can use this integral because the velocity is *always positive* on this interval]

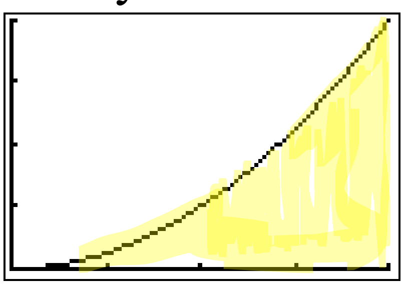
$$\int_0^5 v(t)\ dt = 13.5$$ [the area between the x-axis and the graph]

$\frac{4+5}{2}(3) = \frac{27}{2}$ meters

area of trapezoid

But what if we have a function whose shape is not a figure from a past Geometry class?

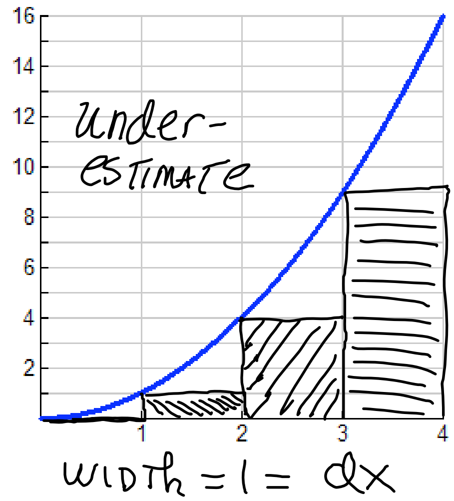
Here is my favorite function $f(x) = x^2$ on [0, 4]



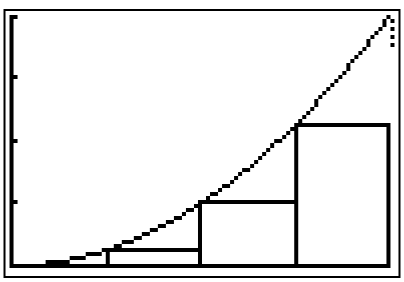
How could we find $\int_0^4 x^2\ dx$?

We could try to fit rectangles under the curve to find an approximation of the area. This is called the Rectangular Approximation Method. Let's try using *four rectangles* and using the left-hand endpoint of each interval to find each height. This will be called the Left Rectangular Approximation Method or LRAM and this is what it looks like.

$\int_0^4 x^2 dx \approx$ LRAM



$$LRAM = 1[f(0)+f(1)+f(2)+f(3)]$$
$$= 1[0+1+4+9]$$
$$= 14$$
$$\int_0^4 x^2\,dx \approx LRAM$$
$$LRAM = 14$$



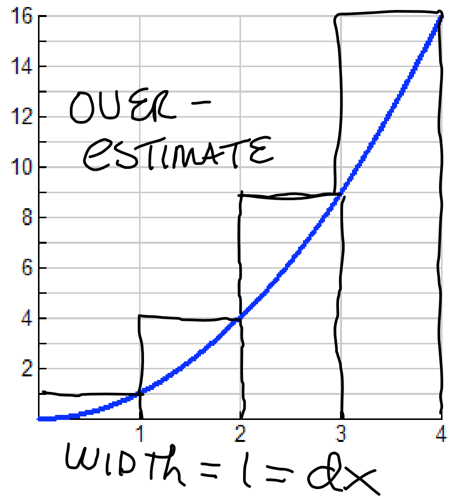
If we divide into four equal subintervals, then there will be four rectangles, If we use the left-hand endpoint of each sub-interval to obtain our height [using the function], then our LRAM = width [$f(0) + f(1) + f(2) + f(3)$]

Which is equal to: LRAM = 1 [$0^2 + 1^2 + 2^2 + 3^2$]

LRAM = 14

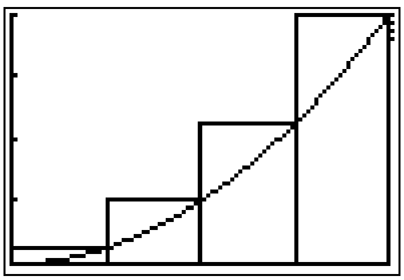
This function is concave up on this interval. Is LRAM an over-approximation or an under-approximation of the actual area?

We could try using the right-hand endpoint to obtain our heights. This is called RRAM and it looks like:



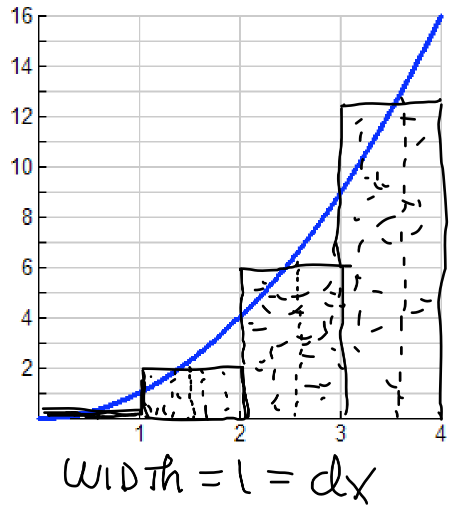
$$\int_0^4 x^2\,dx \approx RRAM$$

$$RRAM = 1\left[f(1)+f(2)+f(3)+f(4)\right]$$
$$= 1\left[1+4+9+16\right]$$
$$= 30$$



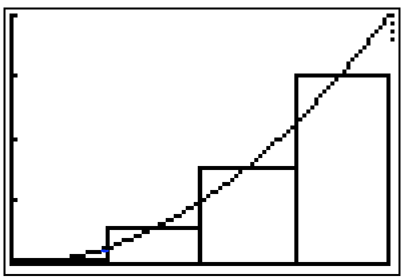
RRAM= width $\left[\,f(1)+\ f(2)+\ f(3)+\ f(4)\right]$
RRAM= $1\left[1^2+2^2+3^2+4^2\right]$
RRAM= 30

Is this an over-estimate or under-estimate?
We could also use the midpoint of each sub-interval to obtain our heights. This is called MRAM and it looks like:



$\int_0^4 x^2 dx \approx$ MRAM

MRAM = 1 [f(.5) + f(1.5) + f(2.5) + f(3.5)]



MRAM= width $\left[f(0.5) + f(1.5) + f(2.5) + f(3.5) \right]$

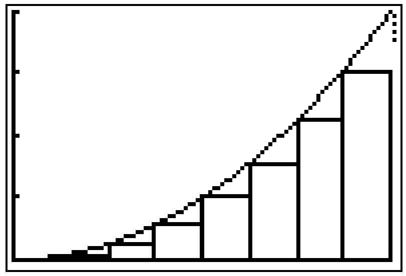
MRAM = 1 $\left[0.5^2 + 1.5^2 + 2.5^2 + 3.5^2 \right]$

MRAM=21

In this case, MRAM is an under-estimate. The actual value of $\int_0^4 x^2 dx = 21\frac{1}{3}$ [We'll learn how to be accurate after we have become experts with RAM.

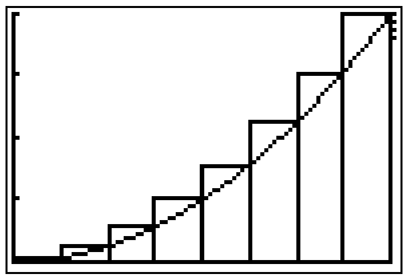
How could we be more accurate using rectangles? *Use more rectangles* which means that our widths would be smaller. Let's see what happens if we use 8 rectangles – which means that our width would be 0.5.

LRAM   =17.5



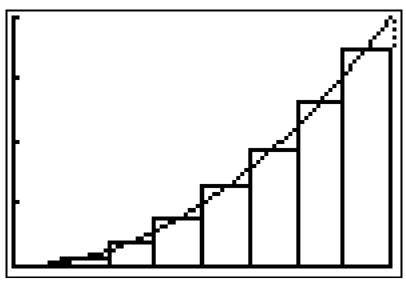
under

RRAM = 25.5



OVER

MRAM = 21.25

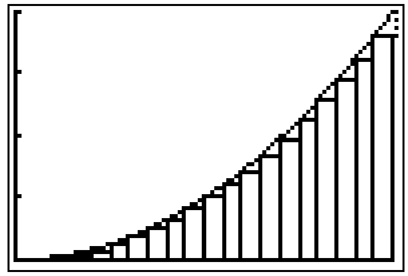


STILL under but close

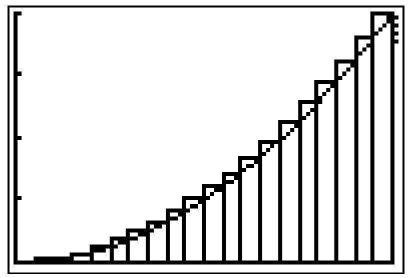
How could we get more accurate? Yes, more rectangles, which would mean that the width would get even smaller.

How about 20 sub-intervals? Our width would be 0.2.

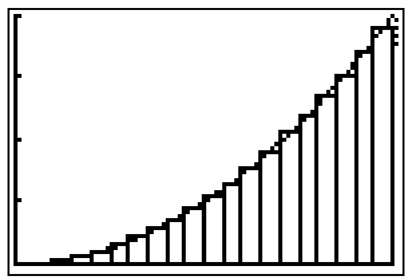
LRAM = 19.76



RRAM=22.96



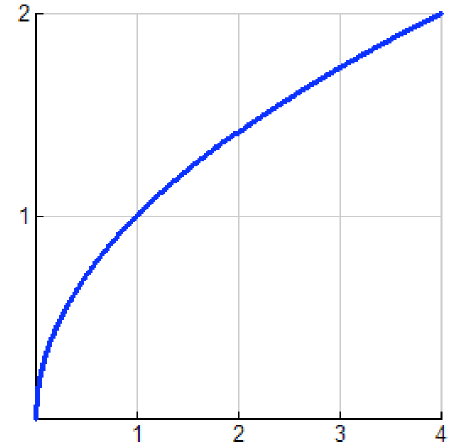
MRAM= 21.32



That's pretty darn close to the actual answer of $21\frac{1}{3}$

For this function, $y = x^2$, which is concave up, LRAM < actual and RRAM > actual.

Not let's consider a concave down function. Let $y = \sqrt{x}$ on [0, 4]

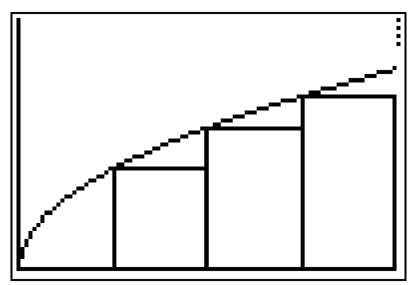


$$\int_0^4 \sqrt{x}\,dx \approx \text{LRAM}$$

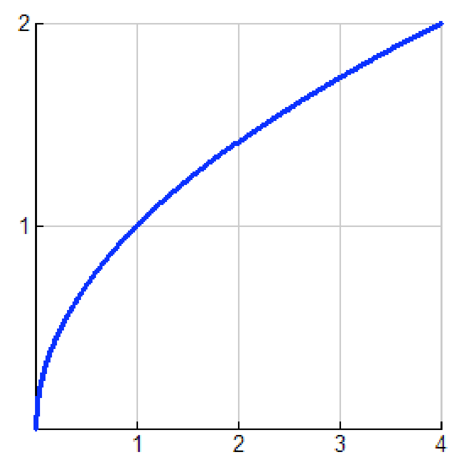
$$\text{LRAM} = 1\,[f(0)+f(1)+f(2)+f(3)]$$

LRAM with four subdivisions
LRAM = width [$f(0) + f(1) + f(2) + f(3)$]
With four subintervals, the width equals 1.

LRAM=4.146

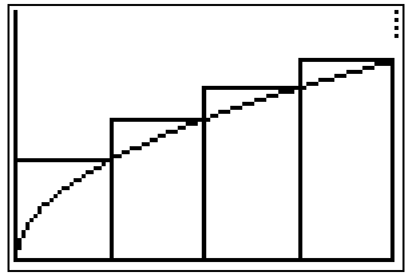


under



$$\int_0^4 \sqrt{x}\,dx \approx \text{RRAM}$$

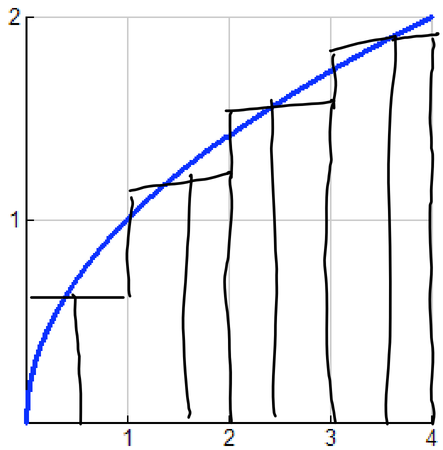
$$\text{RRAM} = 1\,[f(1)+f(2)+f(3)+f(4)]$$



OVER

RRAM=6.146

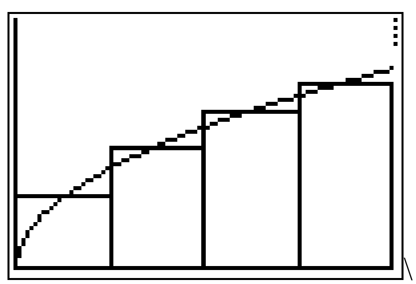
RRAM= 1 [$f(1) + f(2) + f(3) + f(4)$]



$\int_0^4 \sqrt{x}\, dx \approx$ MRAM

$dx = 1 =$ width

MRAM = 1 [$f(.5) + f(1.5) + f(2.5) + f(3.5)$]



OVER

MRAM=5.384

MRAM= 1 [$f(0.5) + f(1.5) + f(2.5) + f(3.5)$]

Actual solution: $\int_0^4 \sqrt{x}\; dx = 5\frac{1}{3}$

This time LRAM was an under-estimate

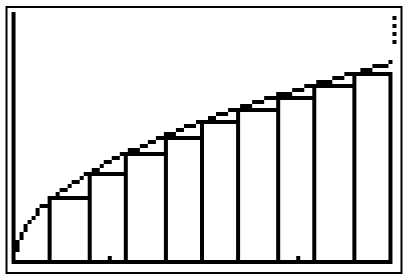
RRAM was an over-estimate

MRAM was an over-estimate but still the closest to the actual value

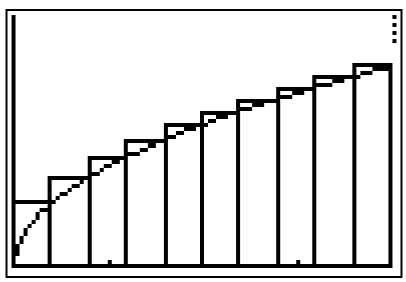
How can we get any closer to the actual answer using RAM?

How about ten subdivisions [or subintervals]?

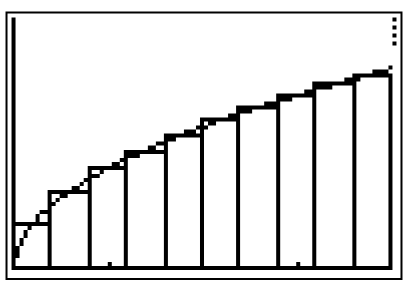
LRAM=4.884



RRAM=5.684



MRAM=5.347



To be accurate we need the width or dx to get very close to zero and the number of rectangles to get very, very large. Hmm! *Sounds like limits to me!*

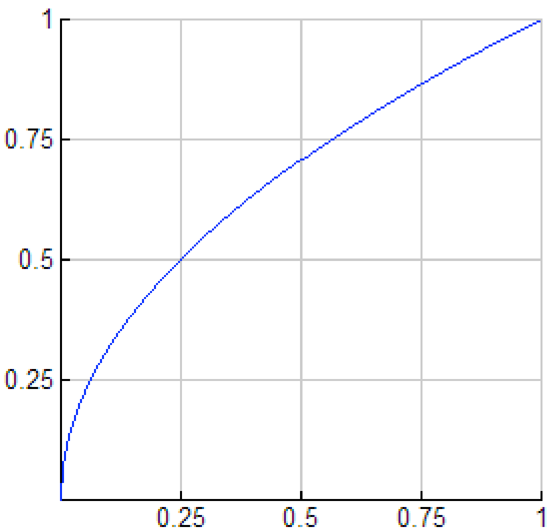
[I told you that limits never go away!]

The process of estimating definite integrals using the sum of the areas of rectangles is called using a Riemann Sum. [The idea is to let the number of rectangles to approach infinity which means the width of the rectangles will approach zero.]

Let's try one. See page 268 #27

$\int_0^1 \sqrt{x}\ dx$ LRAM, then RRAM

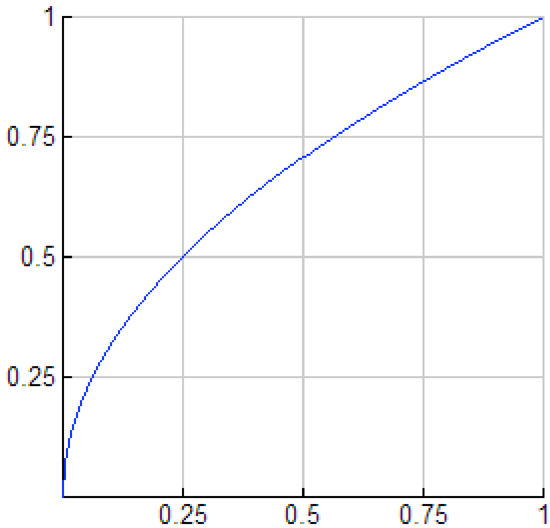
First draw the graph!
Then draw in the rectangles.



$$\int_0^1 \sqrt{x}\ dx \approx LRAM$$

$$LRAM = (0.25)\left(f(0) + f(0.25) + f(0.5) + f(0.75)\right)$$

Now let's try RRAM. Draw in those rectangles!



$$\int_0^1 \sqrt{x}\ dx \approx RRAM$$

$$RRAM = (0.25)\left[f(0.25) + f(0.5) + f(0.75) + f(1) \right]$$

The rate at which water flows into a tank, in gallons per hour, is given by a positive continuous function *R* of time *t*. The table below shows the rate at selected values of *t* for a 12-hour period.

t (hrs)	0	2	4	6	8	10	12
$R(t)$ (gal/hr)	12.5	13.4	13.9	14.3	14.6	14.8	14.7

1. Use a midpoint Riemann sum with three subintervals to approximate:

$$\int_0^{12} R(t)\,dt,$$

Particle *A* moves along a horizontal line with a velocity $v_A(t)$, where $v_A(t)$ is a positive continuous function of *t*. The time t is measured in seconds, and the velocity is measured in cm/sec. The velocity $v_A(t)$ of the particle at selected times is given in the table below.

t (sec)	0	2	5	7	10
$v_A(t)$ (cm/sec)	1.7	6.8	7.4	15.6	24.9

1. Use data from the table to approximate the distance traveled by particle *A* over the interval $0 \le t \le 10$ seconds by using a right Riemann sum with four subintervals. Show the computations that lead to your answer, and indicate units of measure.

Riemann Sums
(sung to the tune of Jingle Bells)

Riemann Sums, Riemann Sums
Counting Areas
Of rectangles whose widths get small
We need to count them all
Riemann Sums, Riemann Sums
Counting Areas
Of rectangles whose widths get small
We need to count them all.

We learn to integrate
It's really lots of fun.
It's easier to find
Than those old Riemann Sums
We learn to sub a u
When things get sort of hard
But most of all we tabulate
When we get sick of parts.
[repeat the refrain]