Riemann Sums
(sung to the tune of Jingle Bells)

Riemann Sums, Riemann Sums
Counting Areas
Of rectangles whose widths get small
We need to count them all
Riemann Sums, Riemann Sums
Counting Areas
Of rectangles whose widths get small
We need to count them all.

We learn to integrate
It's really lots of fun.
It's easier to find
Than those old Riemann Sums
We learn to sub a u
When things get sort of hard
But most of all we tabulate
When we get sick of parts.
[repeat the refrain]

Summary of what we know so far:

We can estimate a definite integral by using a Riemann Sum.

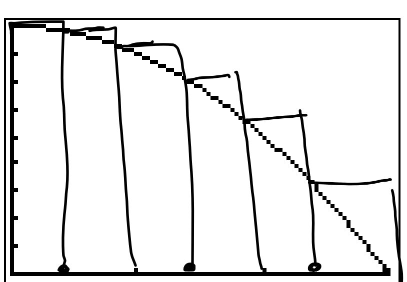
LRAM is the left-hand rectangular approximation
RRAM is the right-hand rectangular approximation
MRAM is the mid-point rectangular approximation

If a function is *increasing and concave up*, then the LRAM is an under-approximation of the actual area; the RRAM is an over-approximation of the actual area; and the MRAM is an under-approximation of the actual area.

If a function is *increasing and concave down*, then the LRAM is an under-approximation of the actual area; the RRAM is an over-approximation of the actual area; and the MRAM is an over-approximation of the actual area.

Now for some new stuff:

Let's consider a decreasing function.
Let $y = 9 - x^2$ on $[0, 3]$



$\frac{1}{2}$

6 Rectangles

$$\int_0^3 (9-x^2)\,dx \approx LRAM$$

Find an approximation of $\int_0^3 \left(9-x^2\right)dx$ using LRAM, RRAM, and MRAM with *six* subintervals

$\int_0^3 \left(9-x^2\right)dx \approx LRAM$

$dx = \text{width} = \frac{1}{2}$

$LRAM = \frac{1}{2}\left[f(0)+f\left(\frac{1}{2}\right)+f(1)+f\left(1\frac{1}{2}\right)+f(2)+f\left(2\frac{1}{2}\right)\right]$

$= \frac{1}{2}\left[9+8.75+8+6.75+5+2.75\right]$

$= 20.125$

OVER

$\int_0^3 \left(9-x^2\right)dx \approx RRAM$

$RRAM = \frac{1}{2}\left[f\left(\frac{1}{2}\right)+f(1)+f(1.5)+f(2)+f(2.5)+f(3)\right]$

$= \frac{1}{2}\left[8.75+8+6.75+5+2.75+0\right]$

under

$= 15.625$

$\int_0^3 \left(9-x^2\right)dx \approx MRAM$

$MRAM = \frac{1}{2}\left[f(.25)+f(.75)+f(1.25)+f(1.75)+f(2.25)+f(2.75)\right]$

$= \frac{1}{2}\left[8.9375+8.4375+7.4375+5.9375+3.9375+1.4375\right]$

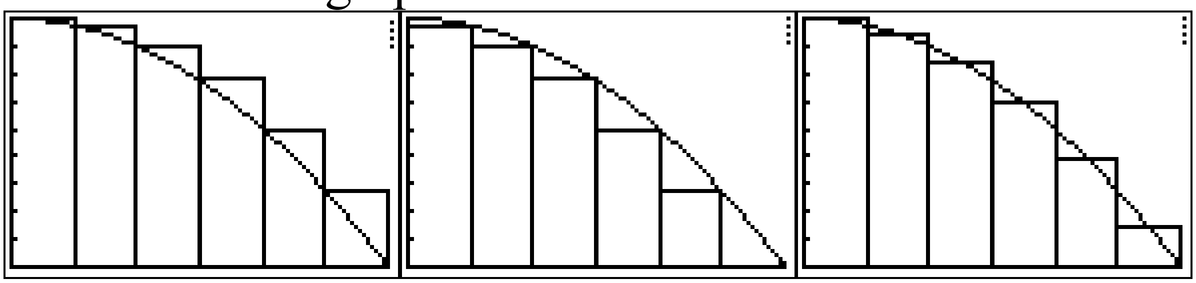
$= 18.0625$

Since $\int_0^3 \left(9 - x^2\right)\, dx = 18$, then

LRAM was an OVER - approximation
RRAM was an UNDER - approximation
MRAM was an OVER - approximation

We should probably remember this for any function that is *decreasing and concave down*.

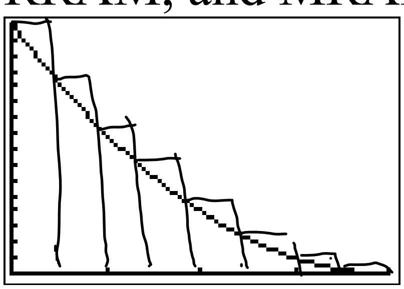
Here are some graphs for our Riemann Sums:



Now let's consider a *decreasing and concave up* function.

Let $f(x) = (x-4)^2$ on [0, 4]

Find an approximation for $\int_0^4 (x-4)^2\, dx$ using LRAM, RRAM, and MRAM using 8 sub-intervals



WIDTH = .5 = dx

$$\int_0^4 (x-4)^2\,dx \approx \text{LRAM}$$

$+ 2$

$$\text{LRAM} = \frac{1}{2}\left[f(0) + f\left(\tfrac{1}{2}\right) + f(1) + f\left(1\tfrac{1}{2}\right) + f(2) + f\left(2\tfrac{1}{2}\right) + f(3) + f\left(3\tfrac{1}{2}\right)\right]$$

$$= \tfrac{1}{2}[16 + 12.25 + 9 + 6.25 + \text{[illegible]} + \quad]$$. over $1 + .25$

$$= 25.5$$

$$\int_0^4 (x-4)^2\,dx \approx \text{RRAM}$$

$$\text{RRAM} = \frac{1}{2}\left[f\left(\tfrac{1}{2}\right) + f(1) + f\left(1\tfrac{1}{2}\right) + f(2) + f(2.5) + f(3) + f(3.5) + f(4)\right]$$

$$= \tfrac{1}{2}[12.25 + 9 + 6.25 + 4 + 2.25 + 1 + .25 + 0]$$

$$= 17.5$$

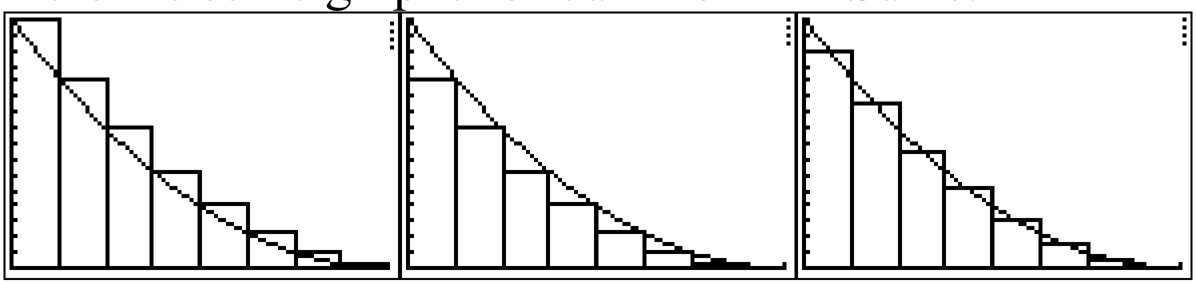
$$\int_0^4 (x-4)^2\,dx \approx \text{MRAM}$$

$$\text{MRAM} = \frac{1}{2}\left[f(. \quad (\text{[illegible]}5) + f(2.25) + f(2.75) + f(3.25) + f(3.75)\right]$$

$$= 21.25$$

Since $\int_0^4 (x-4)^2 \, dx = 21\frac{1}{3}$, then

LRAM was an OVER - approximation
RRAM was an under - approximation
MRAM was an under - approximation
We should probably remember this for any function that is *decreasing and concave up*.
Here are some graphs for our Riemann Sums:



Definition of a Definite Integral

$$\lim_{dx \to 0} \sum_{i=1}^{n} \underbrace{f(c_i)}_{\text{height}} \underbrace{dx}_{\text{WIDTH}} = \int_a^b f(x) \, dx$$

$a \le b$
$a, b \in \mathbb{R}$ FOR NOW

What?!!! [Those limits never go away!]
Think of the rectangles! We are summing up the area of numerous rectangles and we want the width of the rectangles $[dx]$ to get close to zero and the number of rectangles, n, to get close to infinity.

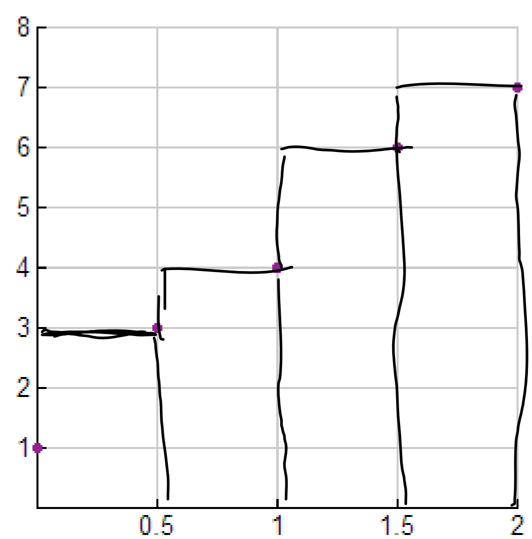
♪ If f is *continuous* on $[a, b]$, then f can be integrated on $[a, b]$.

For now, we are going to concentrate on finding definite integrals using Riemann sums [one of the RAMs].

Let's consider the following problem
The table below contains values of a continuous function f at several inputs of x.

x	0	0.5	1	1.5	2
$f(x)$	1	3	4	6	7

Estimate $\int_0^2 f(x)\,dx$ using a right-hand sum [RRAM] with four equal subintervals, and draw a sketch that illustrates this sum geometrically.



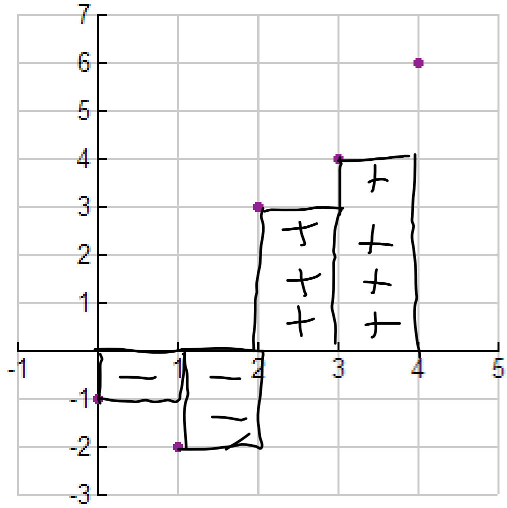
$$\int_0^2 f(x)\,dx \approx RRAM$$

$$RRAM = 0.5\left[f(0.5) + f(1) + f(1.5) + f(2)\right]$$
$$RRAM = 0.5[3+4+6+7]$$
$$RRAM = 0.5[20]$$
$$RRAM = 10$$

We are simply adding up the area of the four rectangles! We need to be careful whenever our $f(x)$ values are negative.

What if we had the following values for $g(x)$ and we want to find $\int_0^4 g(x)\,dx$ using a left-hand sum with four equal subintervals?

x	0	1	2	3	4
$g(x)$	-1	-2	3	4	6



$$\int_0^4 g(x)\ dx \approx \text{LRAM}$$

$$LRAM = (1)\left[g(0)+g(1)+g(2)+g(3)\right]$$
$$LRAM = (1)\left[-1+-2+3+4\right]$$
$$LRAM = 4$$

If we have time:
Let's do some more AP Problems!

The process of estimating definite integrals using the sum of the areas of rectangles is called using a Riemann Sum.

Here is an AP example:

The rate at which water flows into a tank, in gallons per hour, is given by a positive continuous function R of time t. The table below shows the rate at selected values of t for a 12-hour period.

t (hrs)	0	2	4	6	8	10	12
$R(t)$ (gal/hr)	12.5	13.4	13.9	14.3	14.6	14.8	14.7

MRAM

1. Use a midpoint Riemann sum with three subintervals to approximate:

$$\int_0^{12} R(t)\,dt,$$

$$\int_0^{12} R(t)\,dt \approx \text{MRAM}$$

$$\begin{aligned}\text{MRAM} &= 4[R(2) + R(6) + R(10)] \\ &= 4[13.4 + 14.3 + 14.8] \\ &= 170 \text{ gal}\end{aligned}$$

$\int_0^{12} R(t)\,dt$ represents the approx. number of gallons of water which flowed into the tank during $0 \le t \le 12$ hours

Particle A moves along a horizontal line with a velocity $v_A(t)$, where $v_A(t)$ is a positive continuous function of t. The time t is measured in seconds, and the velocity is measured in cm/sec. The velocity $v_A(t)$ of the particle at selected times is given in the table below.

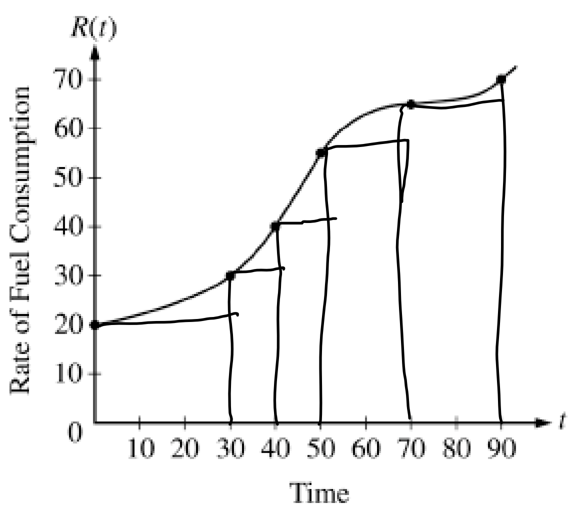
t (sec)	0	2	5	7	10
$v_A(t)$(cm/sec)	1.7	6.8	7.4	15.6	24.9

1. Use data from the table to approximate the distance traveled by particle A over the interval $0 \le t \le 10$ seconds by using a right Riemann sum with four subintervals. Show the computations that lead to your answer, and indicate units of measure.

$$\text{T.D.T.} = \int_0^{10} v(t)\,dt \approx \text{RRAM}$$

$$\text{RRAM} = 2v(2) + 3v(5) + 2v(7) + 3v(10)$$
$$= 141.7 \text{ cm}$$

2003 AP® CALCULUS AB FREE-RESPONSE QUESTIONS



t (minutes)	$R(t)$ (gallons per minute)
0	20
30	30
40	40
50	55
70	65
90	70

3. The rate of fuel consumption, in gallons per minute, recorded during an airplane flight is given by a twice-differentiable and strictly increasing function R of time t. The graph of R and a table of selected values of $R(t)$, for the time interval $0 \le t \le 90$ minutes, are shown above.

(c) Approximate the value of $\int_0^{90} R(t)\,dt$ using a left Riemann sum with the five subintervals indicated by the data in the table. Is this numerical approximation less than the value of $\int_0^{90} R(t)\,dt$? Explain your reasoning.

$$\int_0^{90} R(t)\,dt \approx LRAM$$

$$LRAM = 30R(0) + 10R(30) + 10R(40) + 20R(50) + 20R(70)$$

$$= 30(20) + (10)(30) + (10)(40) + 20(55) + (20)(65)$$

$$= 3700 \text{ gal}$$

This approximation is less because $R(t)$ increases on $0 \le t \le 90$

The rate of fuel consumption is increasing fastest at $t = 45$ minutes. What is the value of $R''(45)$? [Justify]

$R''(45) = 0$ since $R'(t)$ has a maximum at $t = 45$