Secants Versus Tangents

2011ch2daytwo.doc

Mean Value Theorem [another existence theorem]

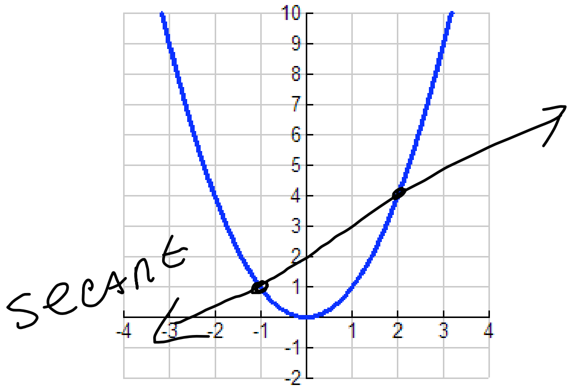
If f is continuous on the closed interval $[a,b]$ AND differentiable on the open interval (a,b), then there exists a number c in (a,b) such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

What's this mean?

If we draw a secant between the endpoints of the closed interval, then somewhere in the interior of the interval the slope of the secant will equal the slope of a tangent line. Or, at some interior point the average rate of change equals the instantaneous rate of change.

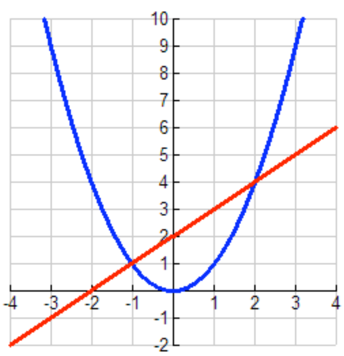
Let's consider my favorite function: $y=x^2$



$$m_{sec} = \frac{f(2) - f(-1)}{2 - (-1)}$$

$m_{sec} = 1$ = A RATE OF Δ

Draw a secant line connecting the points $(-1, 1)$ and $(2, 4)$ then find the equation of this line



Remember: $m_{\text{sec}} =$ *average rate of change [in this case – the average rate of change for the closed interval* $[-1, 2]$

By the MVT there is a c, $-1 < c < 2$, such that $f'(c) = \dfrac{f(2) - f(-1)}{2 - (-1)}$

$$\frac{f(2) - f(-1)}{2 - (-1)} = 1$$

$f(x) = x^2$

$f'(x) = 2x$

let $f'(c) = 1$

$2c = 1$

$c = \frac{1}{2}$

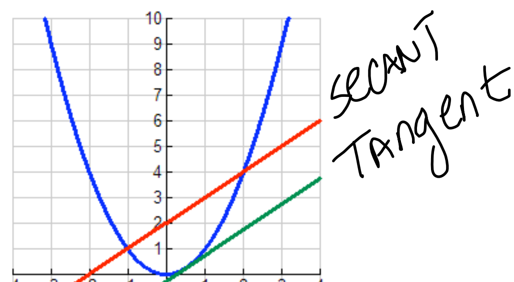
By the Mean Value Theorem, there is a c,

$$-1 < c < 2$$ *, such that*

$$f'(c) = \frac{f(2) - f(-1)}{2 - (-1)}$$

This means that there is some point on the interior of $[-1, 2]$ *where the instantaneous rate of change is equal to the average rate of change. Or there is some point on the interior of* $[-1, 2]$ *where* $m_{\sec} = m_{\tan}$

Find this value of c and use it to draw in the tangent line at that value of c.



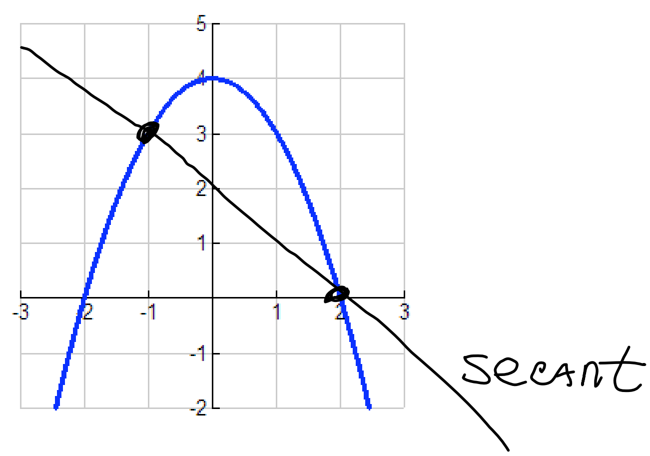
$m_{sec} = m_{Tan}$

Are these two lines parallel? yes

Does the tangent line lie above or below the curve? below

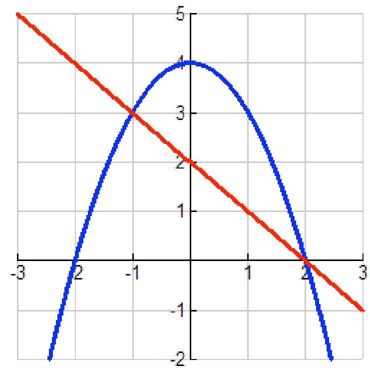
Now let's consider a different function:

$y = 4 - x^2$



Draw a secant line connecting the points $(-1, 3)$ *and* $(2,0)$ *then find the equation of this line*

Remember: $m_{\text{sec}} =$ *average rate of change [in this case – the average rate of change for the closed interval* $[-1, 2]$



By the Mean Value Theorem, there is a c,

-1 < c < 2 , such that $f'(c) = \dfrac{f(2) - f(-1)}{2-(-1)}$

This means that there is some point on the interior of $[-1, 2]$ *where the instantaneous rate of change is equal to the average rate of change. Or there is some point on the interior of* $[-1, 2]$ *where* $m_{\sec} = m_{\tan}$

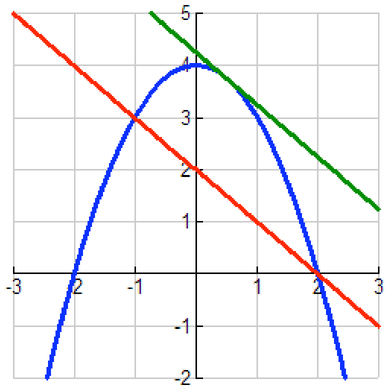
Find this value of c and use it to draw in the tangent line at that value of c.

$$\frac{f(2)-f(-1)}{2-(-1)} = \frac{0-3}{3} = -1 \quad m_{sec}$$

$$f(x) = 4 - x^2 \qquad \text{let } f'(c) = -1$$

$$f'(x) = -2x \qquad -2c = -1$$

$$c = \frac{1}{2}$$



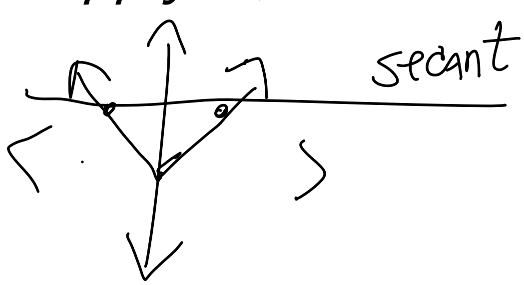
Is the tangent line above or below the curve?

above

Consider the function $y = |x|$ *on* $[-2,2]$

Draw the function and the secant line

Does the MVT apply? NO



$f'(0)$ dne

Let's try some MVT problems

At what value(s) of x does $f(x) = x^3 - x^2 - 2x$ *satisfy the Mean Value Theorem on the interval* $[-1, 1]$

Always start by writing out the MVT statement!!!

By the MVT, there is a c, $-1 < c < 1$,
such that $f'(c) = \dfrac{f(1) - f(-1)}{1 - (-1)}$

$$\frac{f(1) - f(-1)}{1 - (-1)} = \frac{-2 - 0}{2} \qquad m_{sec} = -1$$

$$f'(x) = 3x^2 - 2x - 2$$

let $f'(c) = -1$

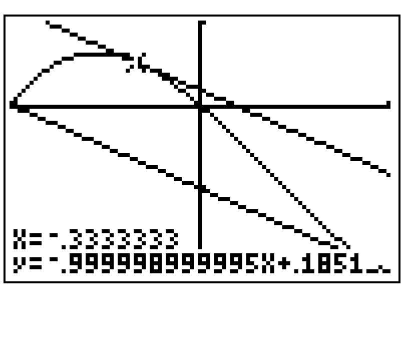
$$-1 = 3c^2 - 2c - 2$$

$$0 = 3c^2 - 2c - 1$$

$$0 = (3c + 1)(c - 1)$$

$c = -\frac{1}{3}$ $\qquad$ $c = 1$

NOT ON $-1 < c < 1$



Notice that where the tan line is above curve

Now find the value that satisfies the MVT for

$y = \frac{x+1}{x}$ *for* $\left[\frac{1}{2}, 2\right]$ $\int$ $y = 1 + \frac{1}{x}$

Always begin by writing out the MVT statement. [yes, always]

By The MVT, There is a c, $\frac{1}{2} < c < 2$, such that $f'(c) = \frac{f(2) - f(\frac{1}{2})}{2 - \frac{1}{2}}$

$\frac{f(2) - f(\frac{1}{2})}{2 - \frac{1}{2}} = \frac{\frac{3}{2} - 3}{\frac{3}{2}}$ m_{sec} or AV RATE OF Δ $= -1$

$f'(x) = -\frac{1}{x^2}$ let $f'(c) = -1$

Need $\frac{1}{2} < c < 2$

$-1 = -\frac{1}{c^2}$

$c = 1$ or $\cancel{c = -1}$

Now for some more interesting problems:
A police officer clocks a commuter's speed at *50 mph as he enters a tunnel whose length is exactly 0.75 miles. A second officer measures* ***the commuter's*** *speed at 45 mph as he exits the tunnel 43 seconds later. Use the MVT to justify the speeding ticket that the commuter received even though the posted speed limit was 50 mph.*

$$\text{AV RATE OF } \Delta = \frac{\Delta \text{ DISTANCE}}{\Delta \text{ TIME}}$$

$$= \frac{0.75 \text{ miles}}{43 \text{ seconds}}$$

$$\text{AV speed} \simeq 62.8 \text{ mph}$$

By the MVT there is a c, $0 < c < 43 \text{sec}$
SUCH THAT INSTANT Speed = AV Speed.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	−4	4	2
4	−1	3	6	7

3. The functions f and g are differentiable for all real numbers, and g is strictly increasing. The table above gives values of the functions and their first derivatives at selected values of x. The function h is given by $h(x) = f(g(x)) - 6$.

(a) Explain why there must be a value r for $1 < r < 3$ such that $h(r) = -5$. IVT

(b) Explain why there must be a value c for $1 < c < 3$ such that $h'(c) = -5$. MVT

By the MVT, there is a c, $1 < c < 3$, such that

$$h'(c) = \frac{h(3) - h(1)}{3 - 1}$$

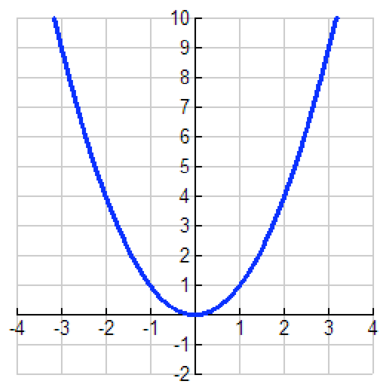
since

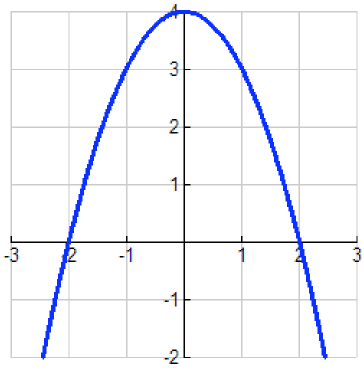
$$\frac{h(3) - h(1)}{3 - 1} = \frac{-7 - 3}{2} = -5$$

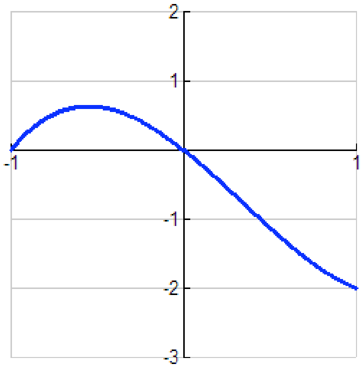
then $h'(c) = -5$ on $1 < c < 3$

Homework: page 177, 178 #39, 41, 43, 57, 67 MUST WRITE the MVT STATEMENT

Handout for Mean Value Theorem







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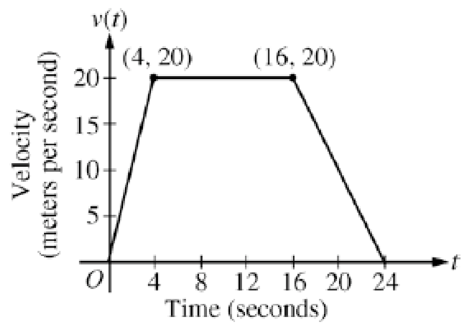

. A car is traveling on a straight road. For $0 \le t \le 24$ seconds, the car's velocity $v(t)$, in meters per second, is modeled by the piecewise-linear function defined by the graph above.

(d) Find the average rate of change of v over the interval $8 \le t \le 20$. Does the Mean Value Theorem guarantee a value of c, for $8 < c < 20$, such that $v'(c)$ is equal to this average rate of change? Why or why not?