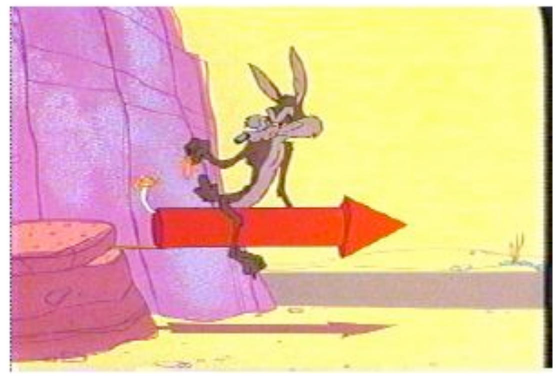
Optimization [also called min/max problems]

A warm-up problem from http://chaoticgolf.com
[Mr. Leckie's awesome website]

Example: Wile E. is after Road Runner again! This time he's got it figured out. Sitting on his ACME rocket he hides behind a hill anxiously awaiting the arrival that "beeping" bird. In his excitement to light the rocket he tips the rocket up. Instead of thrusting himself parallel to the ground where he can catch the Road Runner, he sends himself widely into the air following a path given by the position function

$$s(t) = -.00086x^4 + .067x^3 - 1.67x^2 + 14.77x\,.$$

How high does Wile E. go, and when does he reach that height?



Find the maximum height [using Calculus]

$$s(t) = -0.00086x^4 + 0.067x^3 - 1.67x^2 + 14.77x$$

$$s'(t) = -.00344x^3 + .201x^2 - 3.34x + 14.77$$

$s'(t) = 0$ at $t \approx 7.058$

$t \approx 32.857$

and $s'(t)$ changes from pos to neg values at these times

$s(0) = 0$

$s(7.058) \approx 42.478$ m

$s(32.857) \approx 56.680$ m

max height ≈ 56.680 m

And now on to optimization! [Don't confuse this with related rates]
See Guidelines on page 219

Find two positive numbers such that the product is 192 and the sum is a minimum.

Product $xy = 192$
Sum: $sum = x + y$ which we want to minimize

Primary equation: $x + y = \text{sum}$
Let's call the sum $s(x)$
Secondary equation: $xy = 192$

Rewrite the primary equation using information from the secondary equation so that the primary equation only has one variable.

$xy = 192$

$y = \dfrac{192}{x}$

$s(x) = x + y$

Now, $s(x) = x + \dfrac{192}{x}$

Domain: (0,192]
Now use calculus to find the minimum value of x

$s(x) = x + 192x^{-1}$

$s'(x) = 1 - \dfrac{192}{x^2}$

Note: $s'(x)$ is undefined at $x = 0$ but $x = 0$ is NOT in our domain

Let $s'(x) = 0$

$$0 = 1 - \frac{192}{x^2}$$

$$\frac{192}{x^2} = 1$$

$$192 = x^2$$

$$\sqrt{192} = x$$

$$s''(x) = \frac{384}{x^3}$$

$$s''(\sqrt{192}) > 0$$

Since $s'(\sqrt{192}) = 0$ and $s''(\sqrt{192}) > 0$ then by the Second Derivative Test s has a rel min at $x = \sqrt{192}$

$$\text{minimum sum} = \sqrt{192} + \frac{192}{\sqrt{192}}$$

$$\text{minimum sum} = 2\sqrt{192}$$

Example 2: Find the length and width of a rectangle that has a perimeter of 64 feet and a maximum area.

$P = 2l + 2w$ SECONDARY EQUATION
$A = lw$ PRIMARY EQUATION

$64 = 2l + 2w$
Re-write: $l = 32 - w$

Domain: $(0, 32)$

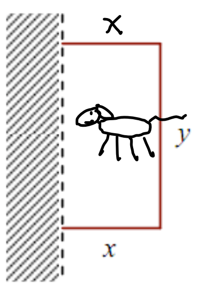
$A(w) = w(32 - w)$
$A(w) = 32w - w^2$
$A'(w) = 32 - 2w$
Let $A'(w) = 0$
Our critical value is $w = 16$

$a''(w) = -2$
$a''(16) = -2$

Since $a'(16) = 0$ and $a''(16) < 0$ then by S.D.T. $a(w)$ has a rel max at $w = 16$, $l = 16$

max area $= (16\text{ ft})^2$
$= 256\text{ ft}^2$

Suppose you want to build a rectangular pen for your dog using a garage wall on one side and a fence on the other three sides:



MAX Area $a = xy$

$2x + y = 40 = P$

$y = 40 - 2x$

If you have 40 feet of fencing available, what should be the dimensions of the pen to yield the largest possible area?

$a(x) = x(40 - 2x)$

Domain: $(0, 20)$

$a(x) = 40x - 2x^2$

$a'(x) = 40 - 4x$

$a'(x) = 0 \quad$ C.V. $x = 10$

$a''(x) = -4$

$a''(10) = -4$

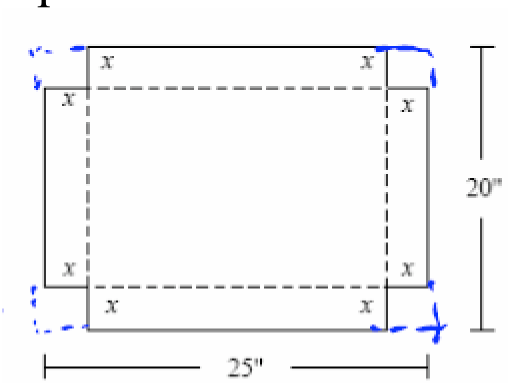
Since $a'(10) = 0$ and $a''(10) < 0$

then by the Second Der Test

$a(x)$ has a rel max at $x = 10$ ft

max area $= 200 \text{ ft}^2$

Let's try this problem from http://chaoticgolf.com
[Mr. Leckie's awesome website]
An open-top box is to be made by cutting congruent squares of side length x from the corners of a 20 – by 25-inch sheet of tin and bending up the sides. How large should the squares be to make the box hold as much as possible?



$V = lwh$

$V = (25-2x)(20-2x)(x)$

Domain: $(0, 10)$

$V(x) = 500x - 90x^2 + 4x^3$

$V'(x) = 500 - 180x + 12x^2$

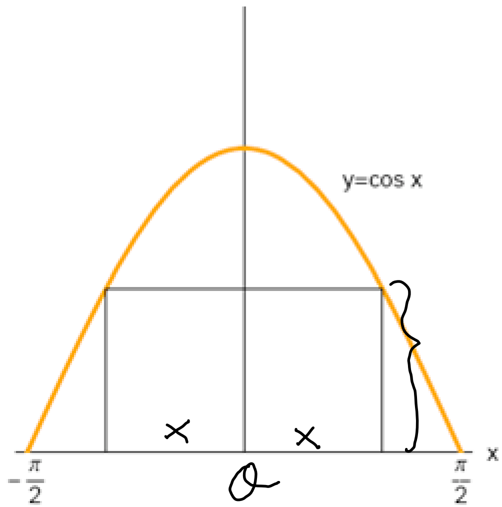
$V'(x) = 0$ at $x \approx 3.681$

at $x \approx 3.681$ $V'(x)$ changes from positive to negative values Hence $V(x)$ has a rel max at $x \approx 3.681$

Max Volume ≈ 820.528 cubic inches

Here's a classic optimization problem:
From:
http://www.frapanthers.com/teachers/zab/APCalculusInaNutshell/ApplicationsDerivatives2004.pdf

A rectangle is to be inscribed under one arch of the cosine curve. What is the largest area the rectangle can have and what dimensions give that area?



$a = lw$

$l = 2x$

$w = \cos x$

$a(x) = \underline{2x}\,\underline{\underline{\cos x}}$

Domain: $(0, \frac{\pi}{2})$

$a'(x) = 2\cos x + -2x\sin x$

at $x \approx .860$ $a'(x)$ changes from positive to negative values. Hence $a(x)$ has a rel max at $x \approx .860$

Our max area ≈ 1.122

You try this problem from http://chaoticgolf.com
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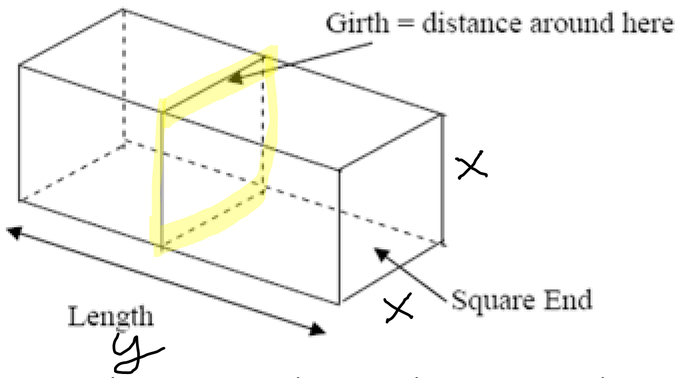
And now, for my favorite of all optimization problems:



A rectangular parcel [package] to be sent by the Royal Mail Service [UK's Postal Service] can have a maximum combined length and girth of 300 cm. Find the dimensions of the parcel of maximum volume that can be sent.
[Assume that the cross section is a square – see above]

$V = lwh$ this is way too many variables

$V = x \cdot x \cdot y$

$V = x^2 y$

$V = x^2(300-4x)$

$V(x) = 300x^2 - 4x^3$

$V'(x) = 600x - 12x^2$

$V''(x) = 600 - 24x$

$V''(50) < 0$

$4x + y = 300$

$y = 300 - 4x$

Domain:
$(0, 75)$

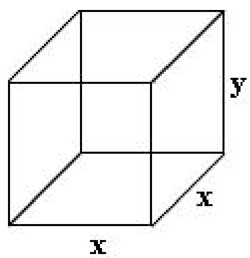
C.V. $x = 50$

since $V'(50) = 0$ and $V''(50) < 0$ then max volume occurs at $x = 50$ cm

max vol $= 250{,}000$ cm^3

Another classic problem:
From:
http://www.mathematicshelpcentral.com/lecture_notes/calculus_1_folder/min-max_and_optimization_problems.htm

A box with a square base with NO top has a surface area of 108 square feet. Find the dimensions that will maximize the volume. [You must use Calculus!]



Primary Equation

$V = lwh$

$V = x^2 y$

$SA = x^2 + 4xy$

$108 = x^2 + 4xy$

Secondary equation

$\frac{108 - x^2}{4x} = y$

Domain: $(0, \sqrt{108})$

$V(x) = x^2 \left(\frac{108 - x^2}{4x} \right)$

$V(x) = 27x - \frac{1}{4}x^3$

$V'(x) = 27 - \frac{3}{4}x^2$

$V'(x) = 0$ at $x = 6$

$V''(x) = -1.5x$ $\quad V''(6) < 0$

Since $V'(6) = 0$, $V''(6) < 0$ then by the SDT $V(x)$ has a rel max at $x = 6$

6 by 6 by 3

max Volume 108 ft^3

Homework: page 223-225 #9, 11, 19, 27, 33 Use your handy-dandy TI to find the critical values and you must show all of your steps like we just did in class. Remember to be mindful of domain issues. Yes, you need justification statements.