EXTREMA* OF A FUNCTION

* plural of extreme

Let f be defined on an interval I containing c

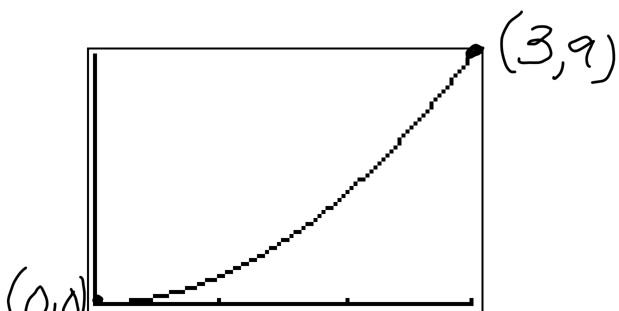
$f(c)$ is the minimum of f on I if $f(c) \le f(x)$ for all x in I [Translation: "the lowest y-value"]

$f(c)$ is the maximum of f on I if $f(c) \ge f(x)$ for all x in I [Translation: "the greatest y-value"]

Extreme Value Theorem [existence theorem]

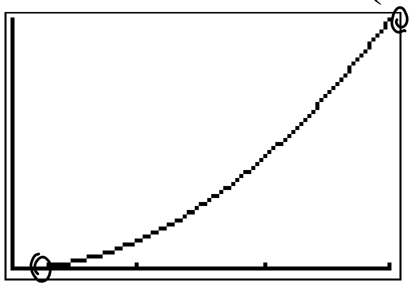
If f is continuous on a closed interval, $[a, b]$, then f has both a minimum and a maximum on the interval

Consider $f(x) = x^2$ on [0, 3]



The absolute minimum is 0 at the point (0, 0)
The absolute maximum is 9 at the point (3, 9)

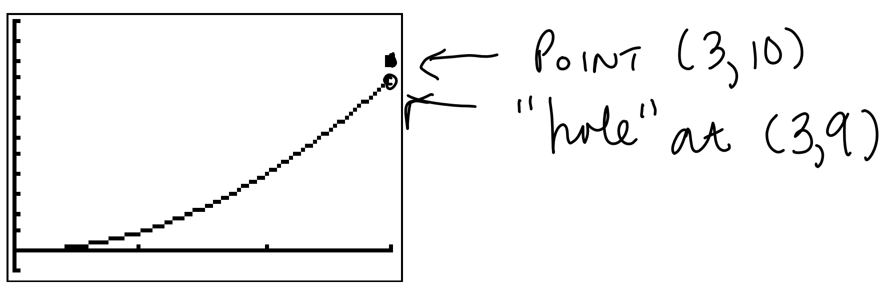
Now consider $f(x) = x^2$ on (0, 3)



OPEN INTERVAL

Since this is an open interval, the endpoints are not included. There is neither a min nor a max, BUT notice that the EVT does not apply because we have an open interval.

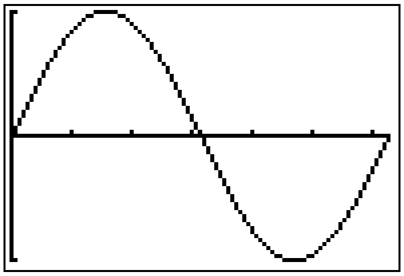
How about $g(x) = \begin{cases} x^2, & 0 \le x < 3 \\ 10, & x = 3 \end{cases}$



EVT does not apply because $g(x)$ is not continuous on [0, 3] but it is defined on [0, 3]

Extrema may occur at endpoints of a closed interval or at an interior point.

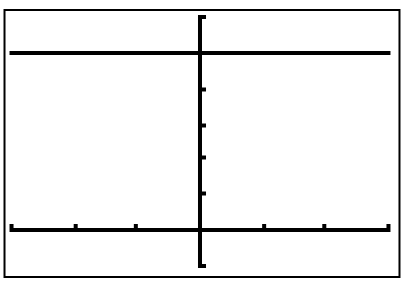
Consider $f(x) = \sin x$ on $[0, 2\pi]$



The absolute minimum value is – 1 at the point $\left(\frac{3\pi}{2}, -1\right)$

The absolute maximum value is 1 at the point $\left(\frac{\pi}{2}, 1\right)$

Consider the function $y = 5$ on $[-3, 3]$



Since the function is continuous on the closed interval, then the EVT must apply. Some people think of this as a weird case of the EVT because the absolute minimum value = absolute maximum value.

Relative [local] versus Absolute [global] extrema

If there is an *open interval* containing c on which $f(c)$ is a maximum, then $f(c)$ is called a relative [local] maximum of f. [Note: Open intervals do not have endpoints]
Who has the relative maximum height in this class?

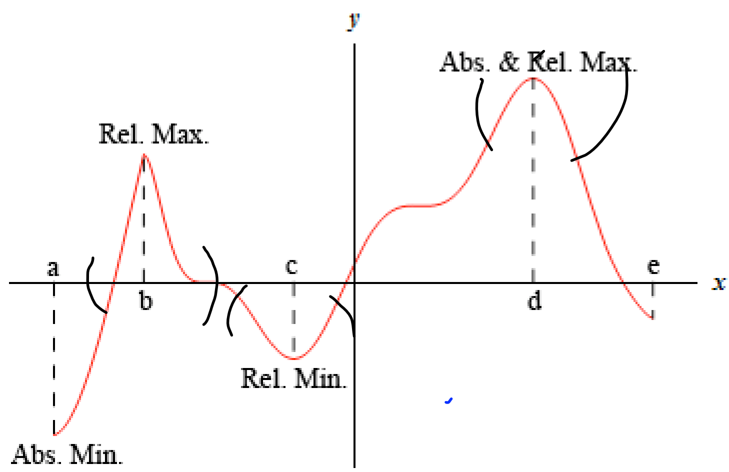
Does this mean that this person has the maximum height in the school? In the world? In the galaxy?

If there is an *open interval* containing c on which $f(c)$ is a minimum, then $f(c)$ is called a relative [local] minimum of f.
Who has the relative minimum age in the middle row?

Does this mean that this person has the minimum age in the school?

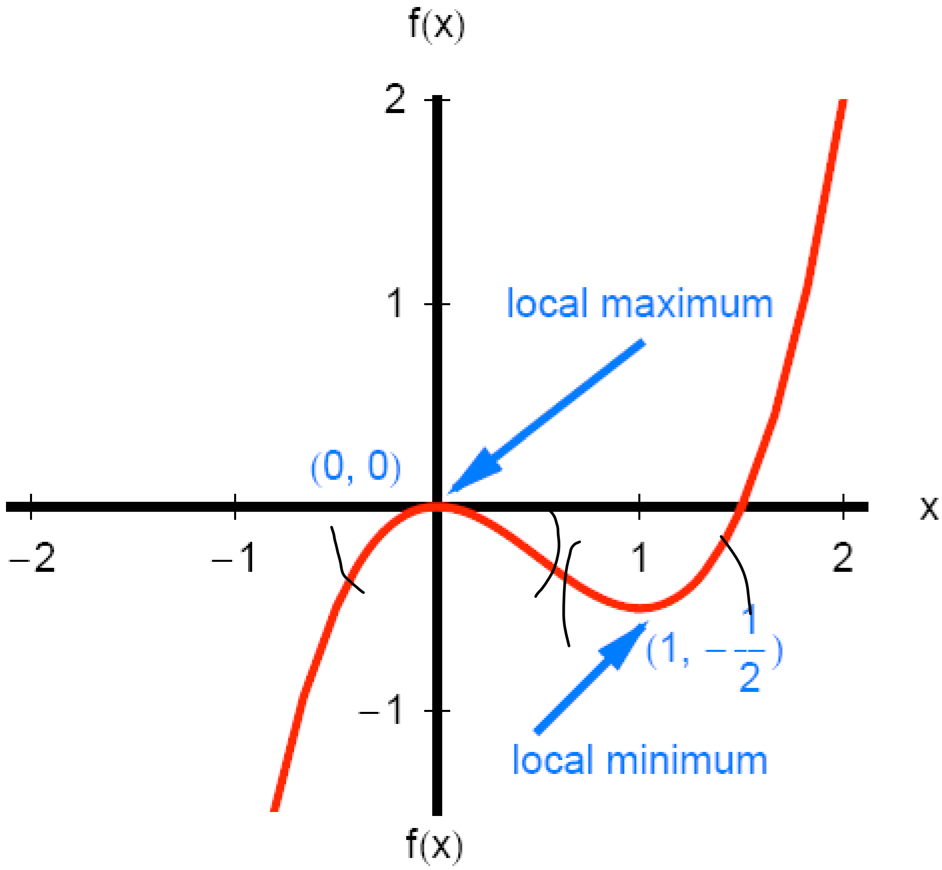
Let's look at this graphically:

From:
http://tutorial.math.lamar.edu/Classes/CalcI/MinMaxValues.aspx



NOTICE THAT THE RELATIVE MAX = ABSOLUTE MAX

From:
http://www.frapanthers.com/teachers/zab/CalculusABReview/ReviewUsingDerivatives.pdf



See page 169 # 1 and # 2

#1 [A, G] BUT NOT CONTINUOUS

a. ☹

b. ab max

c. ☹

d. NOT A MIN OR MAX

e. rel MAX

f. rel min

g. ☹

#2

a. ab min

b. rel max hor tan

c. ☹

d. rel min hor tan

e. rel max NON-DIFF (CUSP)

f. rel min non-DIFF

g. ☹

Critical number(s) or critical value(s)

Let f be defined at c. If $f'(c) = 0$ OR $f'(c)$ is undefined, then c is a critical number of f.

If f has a relative minimum or a relative maximum at $x = c$, then c is a critical number of f. In other words, the critical number(s)/value(s) are worth taking a look at.

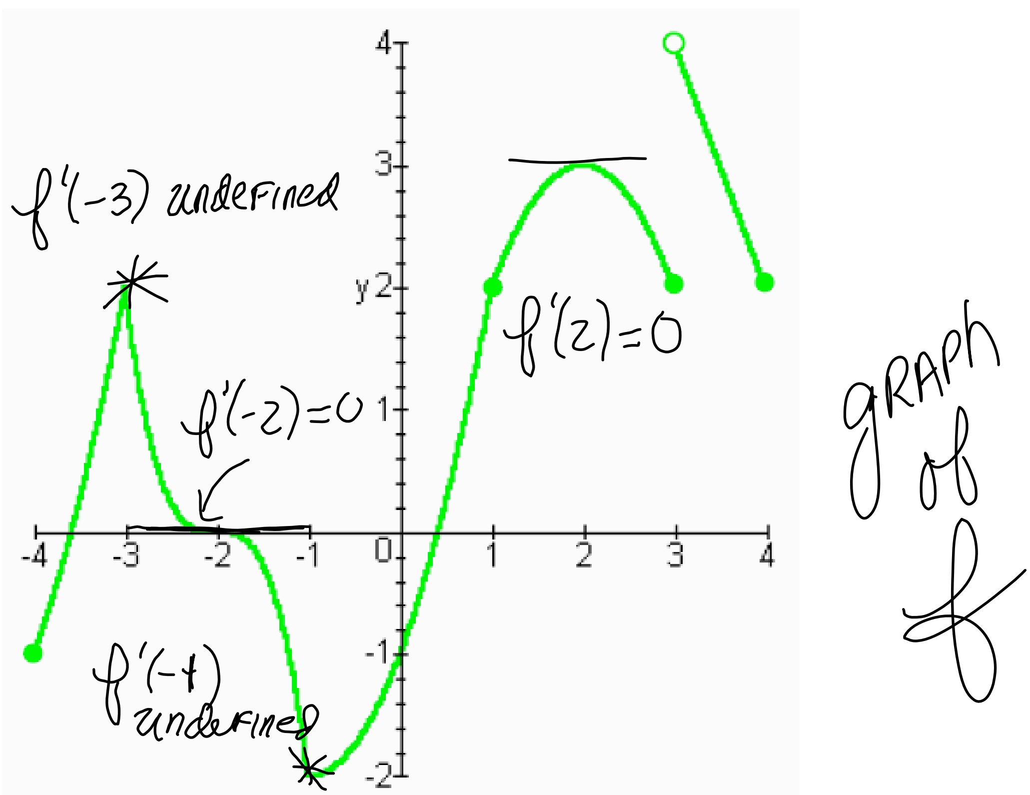
Extrema occur at either endpoints of a closed interval or at critical values of f. If given a closed interval, we must consider the endpoint values.

WARNING!!!!

$f'(c) = 0$ or undefined will give us the *possible* extrema of f. Do not assume that every critical value is an extreme value.

ANOTHER WARNING

You must use calculus to justify an extreme value. You may NOT just draw a graph and label extrema.



The graph above is the graph of f. Does it have any critical values? [In other words, at what values of x is $f'(x)$ either equal to zero or is undefined.]

Now let's do some more problems but without calculators:

Let $f(x) = x^2$ on $[-3, 3]$ This is a continuous function on this interval so EVT will apply.

Step one: find $f'(x)$

$$f'(x) = 2x$$

Step two: find any critical values – in other words, find where the derivative equals zero or is undefined.

$$0 = 2x$$

Critical value: $x = 0$

Step three: evaluate the function at the endpoints and the critical value(s)

$f(0) = 0$ absolute minimum is zero

$f(-3) = 9 = f(3)$ absolute maximum is nine

MUST CHECK endpoints

Let $g(x) = 2x - 1$ on [0, 3] This is a continuous function on this interval so EVT will apply.

$g'(x) = 2$ So there are NO critical values so we only have to check the endpoints

$g(0) = -1$ absolute minimum is -1

$g(3) = 5$ absolute maximum is 5

See guidelines for finding extrema on a closed interval on page 167

Let $h(x)=\sqrt[3]{x}$ on $[-1,1]$ This is a continuous function on this interval so EVT will apply.

$$h'(x)=\frac{1}{3x^{\frac{2}{3}}} \text{ or } \frac{1}{3}x^{\frac{-2}{3}}$$

Critical value at $x=0$ because $h'(0)$ is undefined.

$h(-1)=-1$ absolute minimum

$h(0)=0$

$h(1)=1$ absolute maximum

♪ The critical value did NOT give us an extreme value. But we must always check.

Try the following:

$g(x)=x^{\frac{2}{3}}$ on $[-1, 1]$

$g'(x)=\frac{2}{3}x^{-\frac{1}{3}}$ or $\frac{2}{3\sqrt[3]{x}}$

CRITICAL VALUE at $x=0$

$g(-1)=1$

$g(0)=0$

$g(1)=1$

Hence our ab min value is 0

our ab max value is 1

$$f(x) = \frac{x}{x-2} \text{ on } [3,5]$$

$$f'(x) = \frac{(x-2)(1) - x(1)}{(x-2)^2}$$

$$f'(x) = \frac{-2}{(x-2)^2}$$

C.V. at $x=2$
but NOT in
given domain

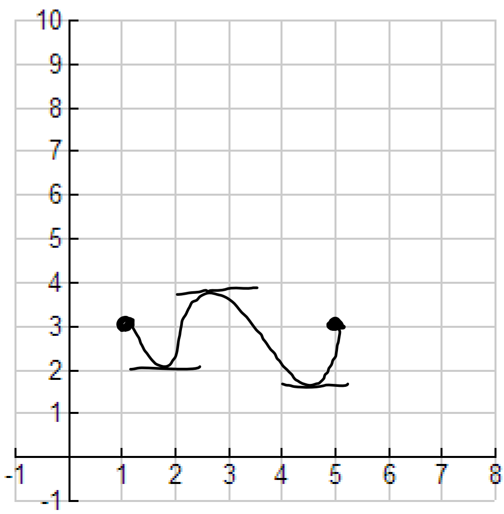
$f(3) = 3$ endpoints AX ab m
ab min

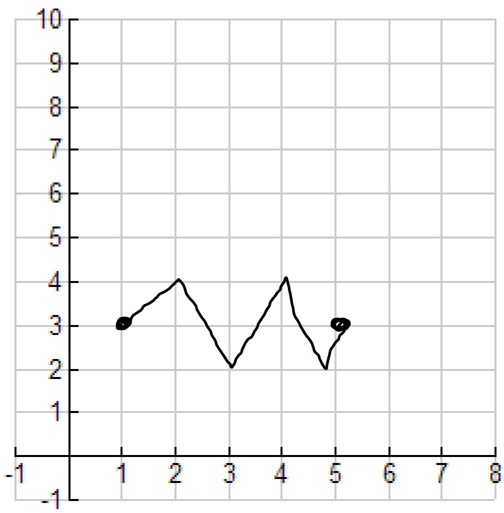
so just check $f(5) =$

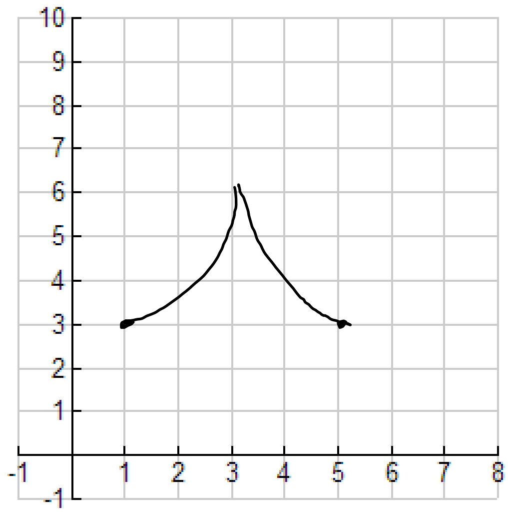
Rolle's Theorem

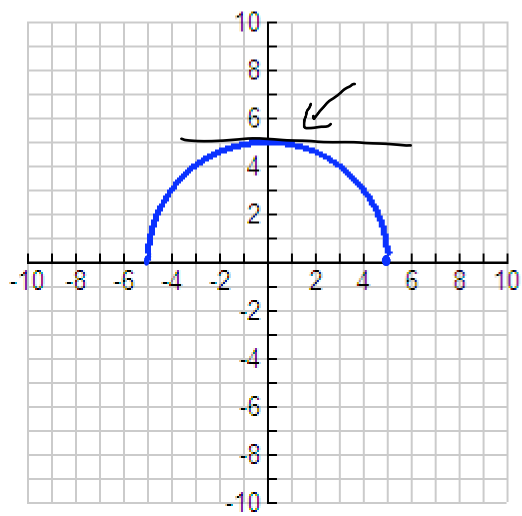
Let's do the Exploration on page 172

Draw a graph that begins and ends at the points (1, 3) and (5, 3). Make sure that your graph is differentiable.









$y = \sqrt{25 - x^2}$

[semi-CIRCLE]

[-5, 5] cont

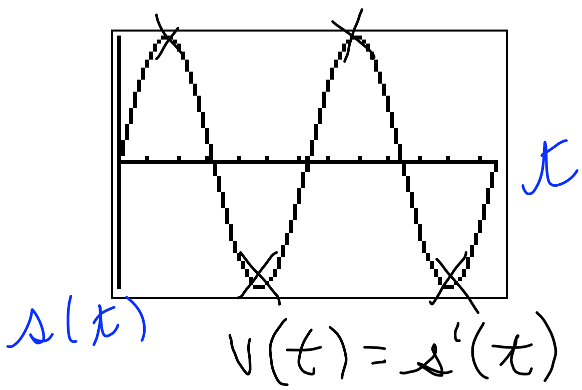
(-5, 5) DIFF

Rolle's Theorem

Let f be continuous on the closed interval $[a, b]$ AND differentiable on the open interval (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) such that $f'(c) = 0$. [Graphically speaking, there is a horizontal tangent somewhere in the interior of the interval.]

In Physics, if you have a position graph for $[a, b]$ which fits the criteria, then you must have a velocity value equal to zero somewhere on the open interval (a, b).

Let this be our position graph



For this particular position graph, how many times will the velocity equal zero on this interval?

Let's consider some functions and see if Rolle's Theorem applies:

$$f(x) = 1 - |x - 1| \text{ on } [0,2]$$

Does $f(0) = f(2)$?

$f(0) = 0$

$f(2) = 0$

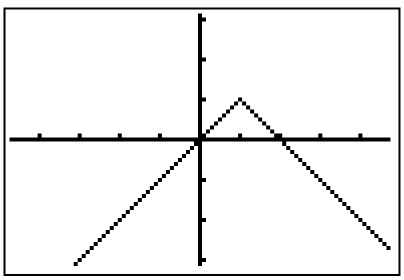
The endpoints have equal y-values but is the function differentiable for all values of x in the interval $0 < x < 2$?

Note: To find $f'(x)$ for any function that involves absolute value, it would be useful to rewrite the function as a piecewise function. In this case:

$$f(x) = \begin{cases} x, & x \le 1 \\ 2 - x, & x > 1 \end{cases} \qquad f'(x) = \begin{cases} 1, & x < 1 \\ -1, & x > 1 \end{cases}$$

That's a big N-O!

Here's the graph: $f'(1)$ dne



Now let's consider $g(x) = \cot\left(\frac{x}{2}\right)$ on $(0, 4\pi)$

Is this function continuous on this interval?
This is a big N-O! So Rolle's Theorem will not apply.
No need to even consider the endpoint values.

How about $h(x) = \frac{1}{|x|}$ on [-1, 1]?

Is this function continuous on this interval? $h(0)$ undefined
Nope! No need to consider Rolle's Theorem or the value of the endpoints.

How about $f(x)=\sqrt{2-x^{\frac{2}{3}}}$ on [-1, 1]?
Is it continuous on the interval? Yes

$f(-1)=1$

$f(1)=1$

So far, so good. We need to know if the function is differentiable on the open interval (-1, 1).

$f'(x)=\frac{1}{2\sqrt{2-x^{\frac{2}{3}}}}\bullet\left(-\frac{2}{3}x^{\frac{-1}{3}}\right)$ which is undefined for x equal to zero. $f'(0)$ is undefined. Once again, Rolle's Theorem does not apply.

Decide if Rolle's Theorem applies and if so, then find all values of c in (a,b) that satisfy Rolle's Theorem [in other words, find any c in the open interval such that $f'(c)=0$]

$f(x)=x^2-5x+4$ on [1, 4]

$f(1)=0$

$f(4)=0$ The endpoint values are equal and no problem with either continuity or differentiability on this interval.

$f'(x)=2x-5$ Set $f'(c)=0$

$0=2c-5$ Hence, $c=\frac{5}{2}$,which is in our interval.

Try: $f(x) = x(x-3)$

Find the two x-intercepts of the function and show that $f'(x) = 0$ at some point between the two x-intercepts. [In other words, that Rolle's Theorem can be applied.]

$f(0) = 0$

$f(3) = 0$

$f(x) = x^2 - 3x$

$f'(x) = 2x - 3$

$0 = -3c2$

on $[0,3]$ $f(x)$ is CONTINUOUS

on $(0,3)$ $f(x)$ is DIFFERENTIABLE

$c = \frac{3}{2}$ WHICH MEANS $f'(\frac{3}{2}) = 0$

Try: $-3x\sqrt{x+1}$ on $[-1,0]$

Determine if Rolle's Theorem can be applied, and if so, then find the value(s) that fulfill Rolle's Theorem.

endpoint CHECK:

$f(-1) = 0$

$f(0) = 0$

$f'(x) = -3\sqrt{x+1} + \frac{-3x}{2\sqrt{x+1}}$

$f'(x) = \frac{-6(x+1) - 3x}{2\sqrt{x+1}}$

NAMELY 6 ___ -9 ___

$f'(x) = \frac{-9x - 6}{2\sqrt{x+1}}$

let $-6 - 9c$ $) = 0$ — c — $\frac{2}{3}$

DIFF C ON (-1,

c o f'(

$-1 < -\frac{2}{3} < 0$ ☺

Homework:

Page 169 #13-18 all

Remember: A critical value is where the first derivative is either *undefined* [but the function is defined] or the first derivative equals *zero*.

AND do the following on page 176 #5, 6, 7, 8

Please clearly write down the function, and its derivative

For the Rolle's Theorem problems show all steps