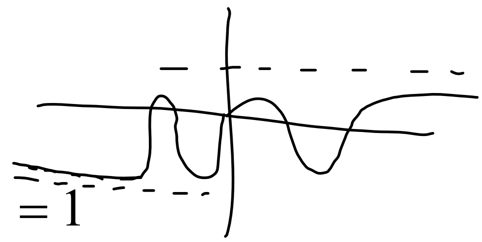
Limits Involving Infinity [∞, not the car]

This is sometimes called “end behavior” of a function

Definition of a horizontal asymptote:
The line $y = L$ is a horizontal asymptote of the graph of f if $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to -\infty} f(x) = L$

♪ ∞ is not a number so *do not say* that $\frac{\infty}{\infty} = 1$



Let’s Explore some more of your favorite - limits!

$$\lim_{x\to\infty} \frac{2x^2 + 200x + 100}{x^2 + 1} = 2$$

used our TI

$$\lim_{x\to\infty} \frac{x-2}{2x+5} =$$

$$\lim_{x\to-\infty} \frac{4x-1}{\sqrt{x^2+2}} = -4$$

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

$$\lim_{x\to\infty} \frac{x^2}{x} = \text{dne}$$

End Behavior

The "easy-squeezy way" to find $\lim\limits_{x \to \pm\infty} f(x)$

This works well when we have a $\dfrac{\textit{polynomial}}{\textit{polynomial}}$ type of function.

To use end behavior method, re-write the rational expression using the term with the greatest degree from both the numerator and the denominator. Cancel if you can because we are considering the limit as x becomes very large and do not have to worry about dividing by zero.

Example:

$$\lim_{x \to \infty} \frac{2x^2 + 200x + 100}{x^2 + 1}$$

The end behavior limit will be: $\lim\limits_{x \to \infty} \dfrac{2\cancel{x^2}}{\cancel{x^2}}$ which equals 2.

A trickier example:

$$\lim_{x \to -\infty} \frac{4x - 1}{\sqrt{x^2 + 1}}$$

$$= \lim_{x \to -\infty} \frac{4x}{-x}$$

$$= -4$$

$$\sqrt{x^2} = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

DO NOT FORGET

The end behavior limit will be: $\lim_{x \to -\infty} \dfrac{4x}{\sqrt{x^2}}$ which will equal

$\lim_{x \to -\infty} \dfrac{4x}{-x}$ [because x is a very large negative number]

Now cancel and we get that the limit is – 4.

See Guidelines page 201

Feel free to use the complicated method used in the textbook but be sure to show all steps.

Let's try some:

$$\lim_{x \to \infty} \frac{5x^2 + 3x + 7}{x + 7}$$

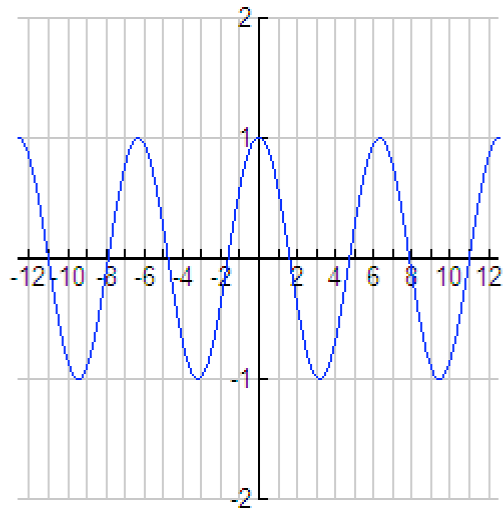
Our end behavior limit will be: $\lim_{x \to \infty} \dfrac{5x^2}{x}$ which equals

$\lim_{x \to \infty} 5x = \infty$

♪ If ∞ is not given as a multiple-choice answer, then choose the "does not exist" option.

Tricky ones:

$f(x) = \cos(x)$



$\lim_{x \to \infty} \cos x = ?$ d.n.e.

Because the graph of $f(x)$ oscillates between $y = -1$ and $y = 1$, the limit does not exist.

Likewise, $\lim_{x \to \infty} \sin x =$ d.n.e.

And the other trig functions will follow [they can all be re-written in terms of $\sin x$ and $\cos x$

Now consider $\lim_{x\to\infty} \frac{\cos x}{x}$

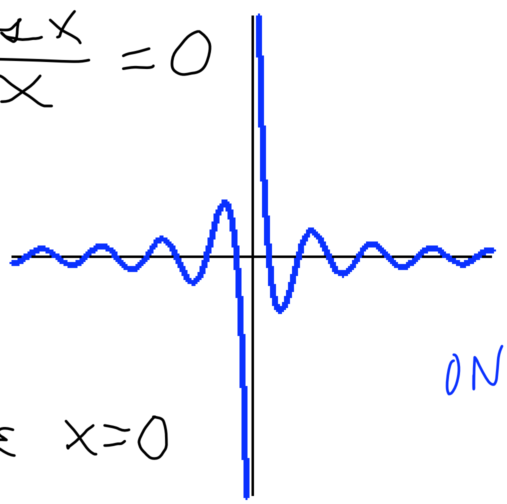
The numerator oscillates between $y=-1$ and $y=1$ BUT the denominator grow larger and larger. Hence,

$\lim_{x\to\infty} \frac{\cos x}{x} = 0$ Hence, $y = 0$ is a horizontal asymptote for $f(x) = \frac{\cos x}{x}$. Notice that the function equals zero and infinite number of times, but we still have a horizontal asymptote.

$$\lim_{x\to\infty} \frac{\cos x}{x} = 0$$

horizontal asymptote $y = 0$

vertical asymptote $x = 0$



$y = \frac{\cos x}{x}$

on $[-8\pi, 8\pi]$

Let's try some: [see graphs on page 205]

$$\lim_{x\to\infty} \frac{3x^2}{x^2+2}$$

$$= \lim_{x\to\infty} \frac{3x^2}{x^2}$$

$$= 3$$

$$\lim_{x \to \infty} \frac{2x}{\sqrt{x^2+2}}$$

$$= \lim_{x \to \infty} \frac{2x}{x}$$

$$= 2$$

since $x > 0$

$\sqrt{x^2} = x$

$$\lim_{x \to -\infty} \frac{2x}{\sqrt{x^2+2}}$$

$$= \lim_{x \to -\infty} \frac{2x}{-x}$$

$$= -2$$

since $x < 0$

$\sqrt{x^2} = -x$

$$\lim_{x \to \infty} \frac{x}{x^2+2}$$

$$= \lim_{x \to \infty} \frac{x}{x^2}$$

$$= \lim_{x \to \infty} \frac{1}{x} = 0$$

All of the rules of limits still apply!

$$\lim_{x\to\infty}\left(2+\frac{x^2}{x^4+1}\right)$$

$$= \lim_{x\to\infty} 2 \quad + \lim_{x\to\infty} \frac{x^2}{x^4}$$

$$= 2 \quad + \lim_{x\to\infty} \frac{1}{x^2}$$

$$= 2$$

$$\lim_{x\to\infty}\frac{4\sin x}{x^2+1}$$

$$= \lim_{x\to\infty} \frac{4\sin x}{x^2}$$

$$= 0$$

4SINX OSCILLATES BETWEEN -4 and 4

$$\lim_{x\to\infty}\frac{2x^2-3x+5}{x^2+1}$$

$$= \lim_{x\to\infty} \frac{2x^2}{x^2}$$

$$= 2$$

$$\lim_{x\to\infty} 4+\frac{3}{x}$$

$$= \lim_{x\to\infty} 4 \quad + \lim_{x\to\infty} \frac{3}{x}$$

$$= 4 + 0$$

$$= 4$$

$$\lim_{x\to-\infty}\left(\frac{x}{2}-\frac{4}{x^2}\right)$$

$$= \lim_{x\to-\infty}\frac{x}{2} - \lim_{x\to-\infty}\frac{4}{x^2}$$

$$= \underbrace{\qquad\qquad}_{dne} - 0$$

$$= d.n.e.$$

Given

$\lim_{x\to 2} f(x) = \infty$ $x=2$ is a VERTICAL ASYMPTOTE

$\lim_{x\to\infty} f(x) = 2$ $y=2$ is a HORIZONTAL ASYMPTOTE

Homework: Page 205 #17, 19, 21, 25, 27, 31, 33
Write the problem and use standard mathematical notation

Show the end behavior like I did in class