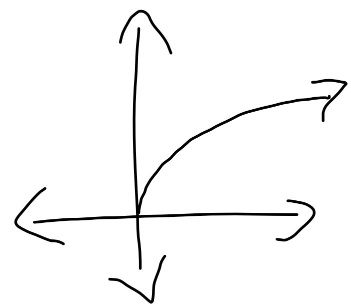
Finding equations of tangent lines

Consider $f(x)=\sqrt{x}$ [domain = ?] $[0,\infty)$

Please find $f'(x)$ AND the equation of the tangent line at the point $(4,2)$



$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x}$$

Eww! Let's multiply by a clever form of one.

$$f'(x)=\lim_{\Delta x\to 0}\frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x}\bullet\frac{\sqrt{x+\Delta x}+\sqrt{x}}{\sqrt{x+\Delta x}+\sqrt{x}}$$

$$=\lim_{\Delta x\to 0}\frac{\cancel{x}+\cancel{\Delta x}-\cancel{x}}{\cancel{\Delta x}\left(\sqrt{x+\Delta x}+\sqrt{x}\right)}$$

$$=\lim_{\Delta x\to 0}\frac{1}{\sqrt{x+\Delta x}+\sqrt{x}}$$

Hence, if $f(x)=\sqrt{x}$, then $f'(x)=\dfrac{1}{2\sqrt{x}}$ or $\frac{1}{2}x^{-\frac{1}{2}}$

Don't forget the domain!

DOMAIN of $f'(x)$: $(0,\infty)$

Now we can easily find the equation of the tangent line when $x=4$.

To write the equation of any line we will need a point and a slope.

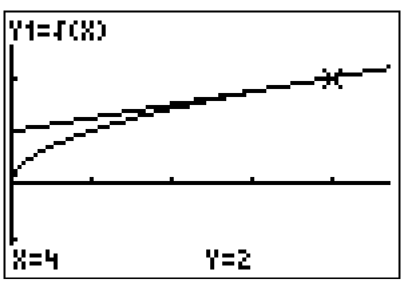
The point can be found by evaluating the function, $f(x)$ at $x = 4$. $f(4) = 2$

The slope of the tangent line can be found by evaluating $f'(x)$ at $x = 4$. $f'(4) = \frac{1}{4}$

And now for the object of our desire – the equation of the tangent line at the point $(4,2)$ will be

$y - 2 = \frac{1}{4}(x - 4)$ $\qquad$ $y - y_1 = m(x - x_1)$

No need to simplify unless you want to. Let's look at what the graph looks like with the tangent line.



Looks good!

Now that we know $f'(x)$, we can find the equation of any tangent line on the curve.

Please find the equation of the tangent line to the curve when $x = 1$ and $x = 9$.

$f(x) = \sqrt{x}$

$f'(x) = \frac{1}{2\sqrt{x}}$

at $x = 1$

Point: $f(1) = 1$

Slope: $f'(1) = \frac{1}{2}$

$y - 1 = \frac{1}{2}(x - 1)$

at $x = 9$

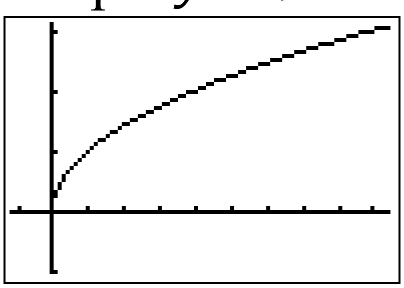
Point: $f(9) = 3$

Slope: $f'(9) = \frac{1}{6}$

$y - 3 = \frac{1}{6}(x - 9)$

♪ You need to be able to find $f'(x)$ analytically. Here's an example of how you might get tricked.

Graph $y = \sqrt{x}$ on [-1.175, 9.4] by [- 1.0333, 3.1]



As $x \to 0$ what is happening to $m_{\tan}$?

Does $f(0)$ exist? $f(0) = 0$

Does $f'(0)$ exist? $\lim_{x \to 0^+} f'(x) = \infty$

$f'(0)$ is undefined

Three forms of the limit definition of derivative

$$f'(c) = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} \quad \text{or} \quad \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

This gives us the $m_{\tan}$ at some value of $x = c, c \in$ Reals

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

This also gives us the $m_{\tan}$ at some value of $x = c, c \in$ Reals

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \quad \text{or} \quad \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This gives us a function rather than a numeric value.

You may use any form and you will need to be able to distinguish between numeric and algebraic answers.

Please find the equation of the tangent line to $f(x) = x^3$ at the point (3, 27)

Since we only need the value of f' at a single point, then I will use one of the $f'(c)$ forms.

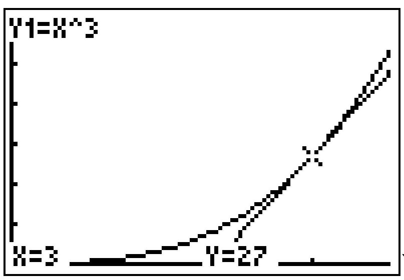
$$m_{\tan}\Big|_{x=3} = f'(3) = \lim_{x\to 3}\frac{x^3-27}{x-3}$$

$$= \lim_{x\to 3}\frac{\cancel{(x-3)}\left(x^2+3x+9\right)}{\cancel{(x-3)}}$$

$$= \lim_{x\to 3}\left(x^2+3x+9\right) = 27$$

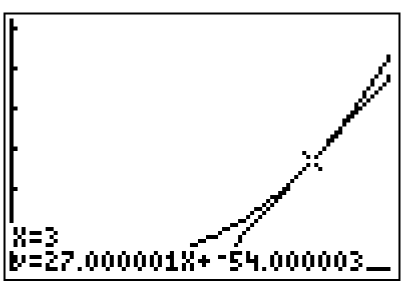
Hence, the equation of the tangent line is

$$y-27 = 27(x-3)$$



Looks good!

By the way, if you try to skip steps and let your TI do your thinking, then you will have an inaccurate tangent line. Here is what TI thinks the equation of the tangent line would be. Notice that it is inaccurate.



Close but not accurate.

To say that a function is differentiable, means that the derivative exists. Just because a function is continuous does not mean that it is differentiable at all points on its domain. BUT IF WE ARE TOLD THAT A FUNCTION IS DIFFERENTIABLE, THEN WE MAY ASSUME THAT THAT FUNCTION IS ALSO CONTINUOUS.

We have already seen that $f(x) = \sqrt{x}$ is continuous on $[0,\infty)$ but not differentiable at $x = 0$.

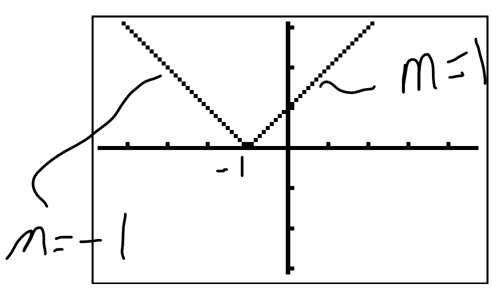
One way to tell if a function is differentiable at a point is to use the second form of $f'(c)$.

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

These must exist AND be equal in order for $f'(c)$ to exist.

Consider $g(x) = |x+1|$

Domain: $(-\infty, \infty)$



graph of $g(x)$ [Let's first draw graph of the derivative]
Does $g'(-1)$ exist?
Let's look at the right- and left-hand limits.

$$\lim_{x\to -1^-} \frac{f(x)-f(-1)}{x-(-1)} \qquad \text{Left-hand limit}$$

$$= \lim_{x\to -1^-} \frac{|x+1|-0}{x+1} = -1$$

[It helps to let x be some number very close to -1 on the left, such as x= -1.001]

$$\lim_{x\to -1^+} \frac{f(x)-f(-1)}{x-(-1)} \qquad \text{Right-hand limit}$$

$$= \lim_{x\to -1^+} \frac{|x+1|-0}{x+1} = 1$$

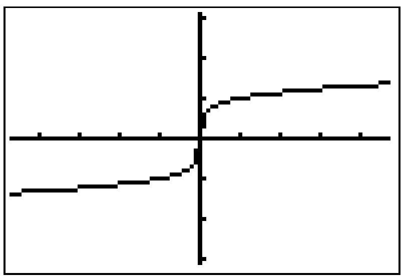
[It helps to let x be some number very close to -1 on the right, such as x= -0.99]

Since the right- and left-hand limits do not agree at x= -1, then $f'(-1)$ does not exist.

Does the derivative exist at $x=2$? yes

At $x = 2$ we can find the instantaneous rate of change by finding the slope of the line. In this case, $f'(2) = 1$

Now consider $h(x) = x^{\frac{1}{5}}$ at $x = 0$



Domain:

Does $h(0)$ exist? Yes, $h(0) = 0$

How about $h'(0)$? $h'(0)$ dne

VERTICAL Tangent

$$\lim_{x \to 0^-} \frac{h(x) - h(0)}{x - 0} = \infty$$

$$\lim_{x \to 0^+} \frac{h(x) - h(0)}{x - 0} = \infty$$

Hence, we have a vertical tangent at $x = 0$, so $h(x)$ is NOT differentiable at $x = 0$

How about $f(x) = \frac{1}{x}$ at $x = 0$?

Does $f(0)$ exist? Nope! So how can we have a tangent line at an undefined point? You can't.

See page 106 #81 through 86.

81 $f'(1)$ dne 82 $f'(-3)$, $f'(3)$ dne 83 $f'(3)$ dne

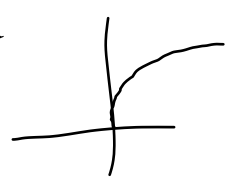
84 $f'(-2)$, $f'(2)$ dne 85 $f'(1)$ dne 86 $f'(0)$ dne

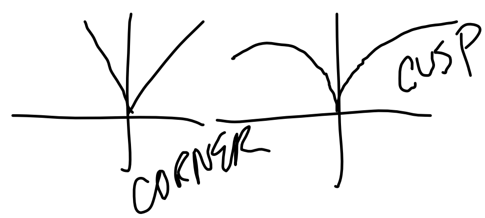
So, based on our detective-like observations, non-differentiable points will occur if we encounter:

(1) discontinuity – no point, no tangent line

(2) sharp corner or cusp – right- and left-hand limits of the derivative will not agree

(3) vertical tangents – slope of a vertical line is undefined

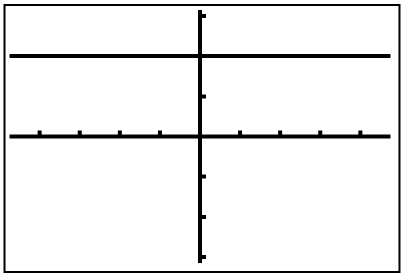




Basic Differentiation Rules

If $f(x) = c$, $c \in$ Reals, then $f'(x) = 0$

In other words, the derivative of a constant equals zero. Why?



$\Delta y = ?$ 0

$f(x) = 11$	$f'(x) =$ 0
$f(x) = \pi$	$f'(x) =$ 0
$f(x) = e$	$f'(x) =$ 0
$f(x) = -711$	$f'(x) =$ 0
$f(x) = (\pi e)^2$	$f'(x) =$ 0

What we know about derivatives so far:

$f(x) = x$ $\qquad f'(x) = 1$

$f(x) = x^2$ $\qquad f'(x) = 2x$

$f(x) = x^3$ $\qquad f'(x) = 3x^2$

$f(x) = \sqrt{x}$ or $x^{\frac{1}{2}}$ $\qquad f'(x) = \dfrac{1}{2\sqrt{x}}$ or $\dfrac{1}{2}x^{\frac{-1}{2}}$

Hmmm! I sense a pattern!

If $f(x) = x^n$, $n \neq 0$, then $f'(x) = nx^{n-1}$

This is called the POWER RULE.

Great! Now we don't have to do the limit definition of derivative. [But, we cannot forget it either.]

Let's find some derivatives:

$f(x)$	$f'(x)$
x^4	$4x^3$
x^5	$5x^4$
x^{100}	$100x^{99}$
x^{-2} OR $\frac{1}{x^2}$	$-2x^{-3}$ or $\frac{-2}{x^3}$
x^{-7} OR $\frac{1}{x^7}$	$-7x^{-8}$ or $\frac{-7}{x^8}$
x^{-100} OR $\frac{1}{x^{100}}$	$-100x^{-101}$

Now finding the equation of tangent lines has become a lot easier.
Please find the equation of the tangent line to $f(x) = \sqrt{x}$ at the point (1, 1).

We need $f'(1)$, the slope of the tangent line at $x=1$

If $f(x) = x^{\frac{1}{2}}$, then $f'(x) = \dfrac{1}{2\sqrt{x}}$ or $\dfrac{1}{2}x^{\frac{-1}{2}}$.

So, $f'(1) = \dfrac{1}{2}$.

Hence, the equation of the tangent line is:

$$y-1=\frac{1}{2}(x-1)$$

Some more differentiation rules

Constant Multiple Rule [think of the CMR for limits]

$$\frac{d}{dx}[c\,f(x)]=c\,f'(x) \text{ where } c\in \text{Reals}$$

CMR in action

$$f(x)=2x^3$$

$$f'(x)=2\bullet\frac{d}{dx}\left(x^3\right)$$

Hence, $f'(x)=2\bullet 3x^2$ or $6x^2$

$$g(x)=7x^{11}$$

$$g'(x)=7\bullet\frac{d}{dx}\left(x^{11}\right)$$

$$g'(x)=7\bullet 11x^{10} \text{ or } 77x^{10}$$

Hmmm! Another pattern?

For $n\neq 0$, $\frac{d}{dx}\left(cx^n\right)=$ cnx^{n-1} $c\in$Reals

$f(x)$	$f'(x)$
$9x^{10}$	$90x^9$
$\frac{-2}{x} = -2x^{-1}$	$2x^{-2}$ or $\frac{2}{x^2}$
$\frac{2x^5}{5}$ or $\frac{2}{5}x^5$	$2x^4$

♪ Whenever you see a grouping symbol, such as the division bar, or radical, it is helpful to re-write the function using rational exponents.

Sum and Difference Rules [think of the limit rules]

$$\frac{d}{dx}\left[f(x) \pm g(x)\right] = f'(x) \pm g'(x)$$

$$\frac{d}{dx}\left[x^3 + x^2\right] = \frac{d}{dx}\left(x^3\right) + \frac{d}{dx}\left(x^2\right)$$

$$= 3x^2 + 2x$$

$$\frac{d}{dx}\left[\sqrt{x} - x^4\right]$$

$$= \frac{d}{dx}\left(x^{\frac{1}{2}}\right) - \frac{d}{dx}\left(x^4\right)$$

$$= \frac{1}{2\sqrt{x}} - 4x^3$$

$$= \frac{1}{2\sqrt{x}} - 4x^3$$

Now let's use all of the new, handy rules:

$f(x)$	$f'(x)$
$3x^2 + 7x$	$6x + 7$
$x^3 + 3x^2 + 3x + 1$	$3x^2 + 6x + 3$
$17 - x^{-2}$	$2x^{-3}$ or $\frac{2}{x^3}$
$\pi x^2 + \sqrt[3]{x}$	$2\pi x + \frac{1}{3}x^{-\frac{2}{3}}$

Sometimes it pays to rewrite your function:

If $y = \frac{7}{11x}$, then $y' = ?$

$y = \frac{7}{11} \cdot \frac{1}{x}$

$y = \frac{7}{11}x^{-1}$ $\qquad$ $y' = \frac{-7}{11}x^{-2}$

If $y = \frac{2}{(3x)^2}$, then $y' = ?$

$y = \frac{2}{9}x^{-2}$ $\qquad$ $y' = \frac{-4}{9}x^{-3}$

If $y=\frac{\sqrt{x}}{x^2}$, then $y'=?$

$$y = x^{-\frac{3}{2}} \qquad y' = -\frac{3}{2}x^{-\frac{5}{2}}$$

If $y=(x+3)^2$, then $y'=?$

$$y = x^2 + 6x + 9$$

$$y' = 2x + 6$$

If $y=\frac{3}{x^{-4}}$, then $y'=?$

$$y = 3x^4$$

$$y' = 12x^3$$

If $y = ex^2$, then $y' = ?$

$y' = 2ex$

Homework: page 104 #25, 26, 27, 28, 29
[Write the function, write the derivative using our new rules, find the specific numerical value of the derivative, then write the equation of the tangent lines]
AND do page 115 #3-17 odds using our new rules [Once again, write the function, then write the derivative]
Please use standard notation!