Chain Rule

Used for finding the derivative of a composite function

If $y = f(u)$ is a differentiable function of u AND $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x and its derivative, $\frac{dy}{dx} = \frac{dy}{du} \bullet \frac{du}{dx}$

Eww! That is not real helpful!

Maybe this is a little bit more helpful:

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \bullet g'(x)$$

This can also be thought of as: Let $g(x) = u$

Then $\frac{d}{dx} f(u) = f'(u)\frac{du}{dx}$

Many people think of this as the derivative of the "outside" times the derivative of the "inside"

Let's see it at work!

Let $y = \sin^3 x$. Let us rewrite this first and then find the derivative.

$$y = (\sin x)^3$$

$$y = u^3$$

Now define what "u" should be.

$$u = \sin x$$

$$\frac{du}{dx} = \cos x$$

$$y = u^3$$

$$y' = 3u^2 \frac{du}{dx}$$

$$y' = 3\sin^2 x \cos x$$

Now let us find the derivative of $y = \sin(11x)$.

Give me a "u"!

$$u = 11x$$

$$\frac{du}{dx} = 11$$

$$y = \sin u$$

$$y' = \cos u \frac{du}{dx}$$

$$y' = 11 \cos(11x)$$

Our goal most of the time will be to rewrite the function in order to see the "inside" or the $g(x)$- part of $f(g(x))$. It is useful to get rid of some grouping symbols such as the division bar or the $\sqrt{}$.

Let $\qquad y=\sqrt{3x^2+17x}$

Rewrite as $\qquad y=(3x^2+17x)^{\frac{1}{2}}$

$y = u^{\frac{1}{2}} \qquad u = 3x^2+17x$

$y' = \frac{1}{2}u^{-\frac{1}{2}}\frac{du}{dx} \qquad \frac{du}{dx} = 6x+17$

$y' = \frac{1}{2\sqrt{3x^2+17x}}(6x+17)$

Let $h(x) = \dfrac{1}{x+13}$

$$h(x) = (x+13)^{-1}$$

$$h(x) = u^{-1}$$

$$h'(x) = -1u^{-2}\frac{du}{dx}$$

$$h'(x) = \frac{-1}{(x+13)^2}$$

$$u = x+13$$

$$\frac{du}{dx} = 1$$

Try: $f(x) = (2x^2+5)^7$

$$f(x) = u^7$$

$$f'(x) = 7u^6\frac{du}{dx}$$

$$f'(x) = 28x(2x^2+5)^6$$

$2x^2 + = \quad 5$

$$\frac{d}{dx} = 4x$$

See page 137 #1-6

$y = f(g(x))$	$u = g(x)$	$y = f(u)$
$y = (6x-5)^4$	$u = 6x-5$ $\frac{du}{dx} = 6$	$y = u^4$ $y' = 24(6x-5)^3$
$y = \frac{1}{\sqrt{x+1}}$	$u = x+1$ $\frac{du}{dx} = 1$	$y = u^{-\frac{1}{2}}$ $y' = -\frac{1}{2}(x+1)^{-\frac{3}{2}}$
$y = \sqrt{x^2-1}$	$u = x^2-1$ $\frac{du}{dx} = 2x$	$y = u^{\frac{1}{2}}$ $y' = \frac{x}{\sqrt{x^2-1}}$
$y = 3\tan(\pi x^2)$	$u = \pi x^2$ $\frac{du}{dx} = 2\pi x$	$y = 3\tan u$ $y' = 6\pi x \sec^2(\pi x^2)$
$y = \csc^3 x$	$u = \csc x$ $du = -\csc x \cot x$	$y = u^3$ $y' = 3u^2 \frac{du}{dx}$ $y' = -3\csc^3 x \cot x$
$y = \cos\left(\frac{3x}{2}\right)$	$u = \frac{3x}{2}$	$y = \cos u$ $y' = -3/2 \sin\left(\frac{3x}{2}\right)$

$\frac{du}{dx} = \frac{3}{2}$

Now find the derivatives of each of these

More of the Chain Rule

Give me a "u"!

Let u be a differentiable function of x

$$\frac{d}{dx} f(u) = f'(u)\frac{du}{dx}$$

Looking for patterns:

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\sin(2x)$ $\quad u = 2x$, $\frac{du}{dx} = 2$	$2\cos(2x)$
$\sin(3x)$	$3\cos(3x)$
$\sin(4x)$	$4\cos(4x)$
$\sin(nx)$, $n \in$ Reals, $n \neq 0$ $\sin u$	$n\cos(nx)$

$$\frac{d}{dx}\cos(nx) = -n\sin(nx)$$

y	y'
$\sqrt{x}$ or $x^{\frac{1}{2}}$	$\frac{1}{2\sqrt{x}}$
$\sqrt{3x}$ or $(3x)^{\frac{1}{2}}$ $u=3x$	$\frac{3}{2\sqrt{3x}}$
$\sqrt{7x}$ or $(7x)^{\frac{1}{2}}$ $u=7x$	$\frac{7}{2\sqrt{7x}}$
$\sqrt{11x}$ or $(11x)^{\frac{1}{2}}$	$\frac{11}{2\sqrt{11x}}$
$\sqrt{g(x)}$ or $(g(x))^{\frac{1}{2}}$ $u=g(x)$ $\frac{du}{dx}$ $g'(x)$	$\frac{1}{2}u^{-\frac{1}{2}}\frac{du}{dx}$ $\frac{g'(x)}{2\sqrt{g(x)}}$
$\sqrt{g(x)+h(x)}$ $u=g(x)+h(x)$ $\frac{du}{dx}=g'(x)+h'(x)$	$\frac{g'(x)+h'(x)}{2\sqrt{g(x)+h(x)}}$
$y=\sqrt{u}$ where u is a differentiable function of x	$\frac{\frac{du}{dx}}{2\sqrt{u}}$ ☆☆☆☆

$g(x)$	$g'(x)$
$\frac{1}{x}$ or x^{-1}	$-1x^{-2}$ or $\frac{-1}{x^2}$
$\frac{1}{2x}$ $(2x)^{-1}$ $u=2x$ $\frac{du}{dx}=2$	$-1u^{-2}\frac{du}{dx}$ $\frac{-1}{(2x)^2}\cdot 2 = \frac{-1}{2x^2}$
$\frac{1}{3x}$ $(3x)^{-1}$ $u=3x$ $\frac{du}{dx}=3$	$-\frac{1}{3x^2}$
$\frac{1}{f(x)}$ $(f(x))^{-1}$ $u=f(x)$ $\frac{du}{dx}=f'(x)$	$-1\,(f(x))^{-2}\,f'(x)$ $= \frac{-f'(x)}{f^2(x)}$
$\frac{1}{f(x)+h(x)}$ $u=f(x)+h(x)$	$\frac{-(f'(x)+h'(x))}{(f(x)+h(x))^2}$
$\frac{1}{u}$ where u is a differentiable function of x	$-\frac{1}{u^2}\frac{du}{dx}$ or $\frac{-\frac{du}{dx}}{u^2}$ ☆☆☆

Sometimes you have to use the Chain Rule more than once. For instance, let $y = \sin^2(3x)$

We could think of this as: $y = (\sin(3x))^2$

If we did, then $y = u^2$ where $u = \sin(3x)$

If $u = \sin(3x)$, then $\frac{du}{dx}$ would need to be found using the Chain Rule. Good thing we already know this.

If $u = \sin(3x)$, then $\frac{du}{dx} = 3\cos(3x)$

Now we can find y' [We can do this!!!]

$y = (\sin(3x))^2$

$y = u^2$

$y' = 2u \bullet \frac{du}{dx}$

$u = \sin(3x)$

$\frac{du}{dx} = 3\cos(3x)$

$y' = 2\underbrace{\sin(3x)}_{u} \bullet \underbrace{3\cos(3x)}_{\frac{du}{dx}}$

$y' = 6\sin(3x)\cos(3x)$

Now let us consider $y = \sqrt{\tan(4x)}$

Stupid $\sqrt{}$ Let us rewrite

$$y = \left(\tan(4x)\right)^{\frac{1}{2}}$$

$$u = \tan 4x$$

$$y = u^{\frac{1}{2}}$$

$$\frac{du}{dx} = 4\sec^2(4x)$$

$$y' = \frac{\frac{du}{dx}}{2\sqrt{u}}$$

$$y' = \frac{2\sec^2(4x)}{\sqrt{\tan(4x)}}$$

Beware of poor reading skills! [When would we need the Chain Rule to find y' ?]

$y = \cos 3x^2$ is read as $y = \cos\left(3x^2\right)$

$y = \left(\cos 3\right)x^2$ is read as $y = \left(\cos 3\right) \bullet x^2$ where cos3 is a constant

$y = \cos\left(3x\right)^2$ is read as $y = \cos\left(9x^2\right)$

$y = \cos^2 x$ is read as $y = \left(\cos x\right)^2$

$y = \cos^2\left(3x^2\right)$ is read as $y = \left[\cos\left(3x^2\right)\right]^2$

$y = \sqrt{\cos x}$ can be read as $y = \left(\cos x\right)^{\frac{1}{2}}$

Try:

Let $f(x) = \sin(\sqrt{x})$. Find $f'(x)$

$$u = \sqrt{x}$$

$$f(x) = \sin u$$

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$$

$$f'(x) = \cos u \frac{du}{dx}$$

$$f'(x) = \frac{\cos(\sqrt{x})}{2\sqrt{x}}$$

Let $g(x) = \tan^2(x^2)$. Find $g'(x)$

$$u = \tan(x^2)$$

$$g(x) = u^2$$

$$g'(x) = 2u \frac{du}{dx}$$

$$\frac{du}{dx} = 2x\sec^2(x^2)$$

$$= 4x\sec^2(x^2)\tan(x^2)$$

Let $h(x) = \sqrt{\sin(2x)}$. Find $h'(x)$

$$h(x) = \sqrt{u}$$

$$u = \sin(2x)$$

$$h'(x) = \frac{1}{2\sqrt{u}} \frac{du}{dx}$$

$$\frac{du}{dx} = 2\cos(2x)$$

$$h'(x) = \frac{\cancel{2}\cos(2x)}{\cancel{2}\sqrt{\sin(2x)}}$$

Homework: Read 2.4, do page 137 #7, 11, 13, 17, 21 25, 26, 27 41, 43, 45, 47, 51, 53 [Please state your u and $\frac{du}{dx}$ and clearly show your steps using standard mathematical notation, blah, blah, blah, …]

If all you have is the function and the derivative, then NO points will be awarded.

Note: Some problems requires the Chain Rule AND the Product Rule, or they require the Chain Rule AND the Quotient Rule