Implicit Differentiation

Used for relations such as circles, ellipses, etc. We will imply that y depends on x or that some specified variable depends on another specified variable. We will also use this for "related rates" problems which we will conquer tomorrow.

$\frac{d}{dx}x=1$ We know this. It is not difficult. Notice that we took the derivative in terms of x and that our variables matched.

But what if we have $\frac{d}{dx}y$? That's different. We need to imply that y depends on x.   And thus get that

$\frac{d}{dx}y=\frac{dy}{dx}$ ♫♪ You need to memorize this!!!!

$\frac{d}{dy}y=1$ $\frac{d}{dt}y=\frac{dy}{dt}$

$\frac{d}{dx}x^2=2x$ Not so bad. Notice that the variables match.

$\frac{d}{dx} y^2 = 2y\frac{dy}{dx}$ Yes, you must include the $\frac{dy}{dx}$ because that will imply that y depends on x.

$\frac{d}{dx} x^{711} =$ $711x^{710}$

$\frac{d}{dx} y^{711} =$ $711y^{710}\frac{dy}{dx}$

$\frac{d}{dx} \tan x =$ $\sec^2 x$

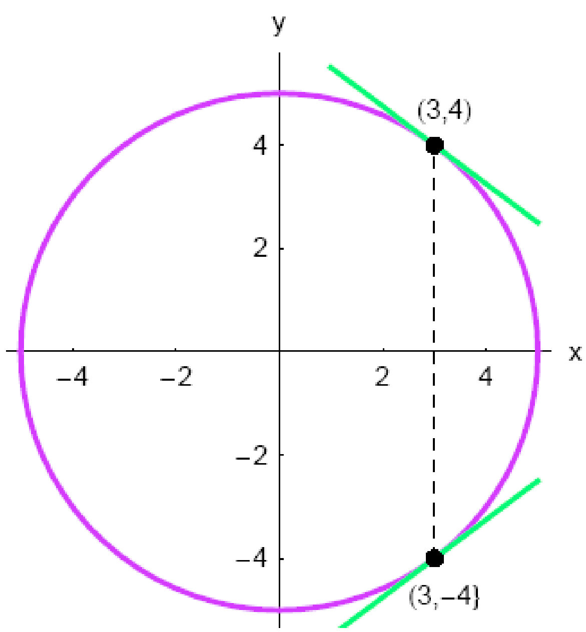
$\frac{d}{dx} \tan y =$ $\sec^2 y \frac{dy}{dx}$

Now that we can do implicit differentiation, we can see it in action.

The following example is from Mr. Zab's webpage at:
http://www.frapanthers.com/teachers/zab/APCalculusInaNutshell/ImplicitDifferentiation2004.pdf

Example 1: Find the slope of the tangent line to any point on the graph of the circle defined *implictly* as $x^2 + y^2 = 25$.

Solution: First look at the graph below.



This is the graph of $x^2 + y^2 = 25$ which the tangent lines drawn in at the points (3, - 4) and (3, 4)
Notice that $x^2 + y^2 = 25$ is a relation, not a function so we must imply that y depends on x. Why don't we find the equation of tangent lines at the indicated points?

First we must find the slopes of the tangent lines, so we need to take the derivative. Oops! Looks like we two different variables, so we'll need to use implicit differentiation.

$$x^2 + y^2 = 25 \qquad \text{Take } \frac{d}{dx} \text{ of each term}$$

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}25$$

$$\cancel{2}x + \cancel{2}y\frac{dy}{dx} = 0$$

$$y\frac{dy}{dx} = -x$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

Now we can find the values we need in order to write the equation of the tangent lines.

$$\left.\frac{dy}{dx}\right|_{(3,-4)} = \frac{-3}{-4} \qquad \text{AND} \qquad \left.\frac{dy}{dx}\right|_{(3,4)} = \frac{-3}{4}$$

(x, y) $\qquad$ (x, y)

So, the equations of the tangent lines are:

$y + 4 = \frac{3}{4}(x - 3)$ at $(3, -4)$

$y - 4 = -\frac{3}{4}(x - 3)$ at $(3, 4)$

$y - y_1 = m(x - x_1)$

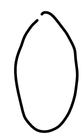
Now let's find any horizontal or vertical tangents to the circle!

We know that: $\dfrac{dy}{dx} = \dfrac{-x}{y}$

Horizontal Tangent if

if $x = 0$

Points $(0,5), (0,-5)$



$-\dfrac{x}{y} = 0$

VERTICAL Tangent if $\frac{dy}{dx}$ is undefined

$\frac{dy}{dx}$ is undefined

if $y = 0$

Points: $(-5, 0), (5, 0)$

$$\frac{d}{dx}(xy) = ?$$

Eeek! This will need implicit differentiation and we must be sure that we use our friend the Product Rule.

$$\frac{d}{dx}(xy) = \left(\frac{d}{dx}x\right) \bullet y + \left(\frac{d}{dx}y\right) \bullet x$$

$$\mathrm{I}'\,\mathrm{II} + \mathrm{II}'\,\mathrm{I}$$

$$= y + x\frac{dy}{dx}$$

We must make sure that we keep a "constant vigilance" for these tricky relations!

Here is what "Mr. Zab" say at this webpage:

Implicit Differentiation Process

1. Differentiate both sides of the equation with respect to x.
2. Collect the terms with dy/dx on one side of the equation.
3. Factor out dy/dx.
4. Solve for dy/dx.

(1) $\frac{d}{dx}\cos(xy)$

$= \frac{d}{dx}\cos u$

$= -\sin u \frac{du}{dx}$

$= [-\sin(xy)][y + x\frac{dy}{dx}]$

$u = \underline{x}\,\underline{y}$

$\frac{du}{dx} = y + x\frac{dy}{dx}$

(2) $\frac{d}{dx}\sqrt{xy}$

$= \frac{d}{dx}\sqrt{u}$

$= \frac{\frac{du}{dx}}{2\sqrt{u}}$

$= \frac{y + x\frac{dy}{dx}}{2\sqrt{xy}}$

$u = xy$

$\frac{du}{dx} = y + x\frac{dy}{dx}$

(3) $\frac{d}{dx}\underline{\sin y}\,\underline{\cos x}$

$= \underbrace{\cos y \frac{dy}{dx}}_{\frac{d}{dx}\sin y} \cos x - \sin x \sin y$

Now let's find the equation of the line tangent to the curve $x^2y + 3x = y^2 + 1$ at the point $(1, -1)$

TO FIND SLOPE OF TAN LINE FIND $\frac{dy}{dx}$

$\frac{d}{dx}\underline{x^2}\,\underline{y} + \frac{d}{dx}3x = \frac{d}{dx}y^2 + \frac{d}{dx}1$

$2xy + x^2\frac{dy}{dx} + 3 = 2y\frac{dy}{dx}$

$2xy + 3 = \frac{dy}{dx}(2y - x^2)$

$-\frac{2xy+3}{2y-x^2}$

$\left.\frac{dy}{dx}\right|_{\substack{(1,-1)\\(x,y)}} = \frac{-2+3}{-2-1} = -\frac{1}{3}$

$$y+1=-\frac{1}{3}(x-1)$$

Homework: do page 146 #1, 5, 11, 27, 35
We will practice some actual AP problems later this week.