RELATED RATES
AN APPLICATION OF IMPLICIT DIFFERENTIATION

[Doctor Who – time lord]

Many things change with time. Our goal is to find the rate at which some quantity is changing by relating the quantity to other quantities whose rates of change are known. **Blah, blah, …**

Short version: we will need to take $\frac{d}{dt}$ of some equations to find some missing rate.

Warm-up:

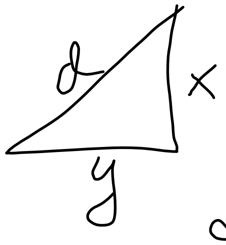
Find $\frac{d}{dt}$ of the following equations:

$$A = s^2$$

$$\frac{d}{dt} A = \frac{d}{dt} s^2$$

$$\frac{dA}{dt} = 2s \frac{ds}{dt}$$

$$x^2 + y^2 = d^2$$



$$\frac{d}{dt}x^2 + \frac{d}{dt}y^2 = \frac{d}{dt}d^2$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2d\frac{dd}{dt}$$

$$S = 2\pi\,\underline{r}\,\underline{\underline{h}}$$

$$\frac{d}{dt}S = 2\pi\left[\frac{d}{dt}\,\underline{r}\,\underline{\underline{h}}\right]$$

$$\frac{dS}{dt} = 2\pi\left[h\frac{dr}{dt} + r\frac{dh}{dt}\right]$$

Consider the area of a rectangle. How is the area changing at the moment that the length is 15, the width is 10, and the length is growing at a rate of 3 inches/hour while the width is decreasing at a rate of 3 inches/hour?

Find $\frac{dA}{dt}$ when, $l=15$,
$W=10$

$$a = lw$$

$$\frac{d}{dt}a = \frac{d}{dt}\,\underline{l}\,\underline{\underline{w}}$$

$$\frac{da}{dt} = w\frac{dl}{dt} + l\frac{dw}{dt}$$

$$\frac{dl}{dt} = \frac{3\text{in.}}{\text{hr}} \qquad \frac{dW}{dt} = \frac{-3\text{in.}}{\text{hr}}$$

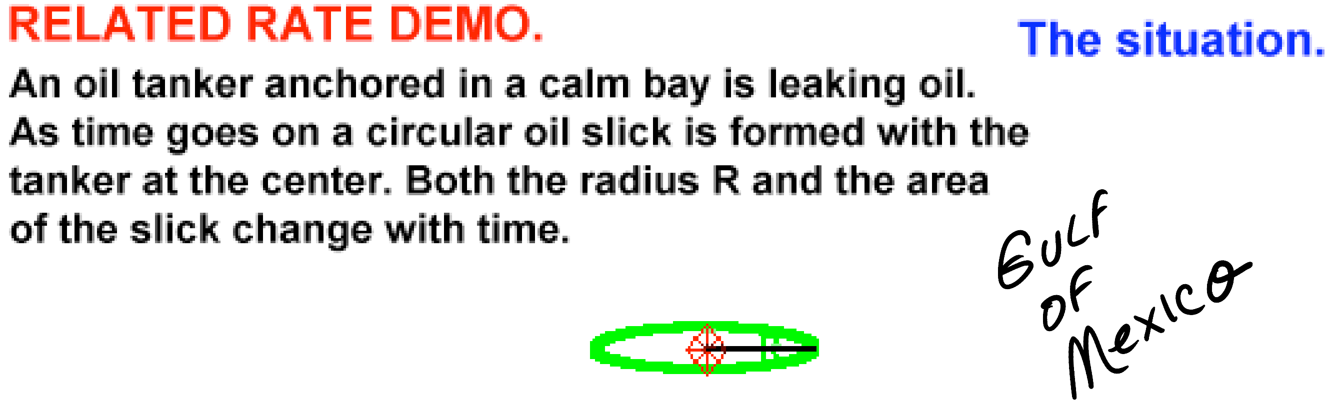
$$\frac{da}{dt} = (10)(3) + (15)(-3)$$

$$= 30\,\frac{\text{in}^2}{\text{hr}} - 45\,\frac{\text{in}^2}{\text{hr}}$$

$$\frac{da}{dt} = -15\,\frac{\text{in}^2}{\text{hr}}$$

All related rates problems involve something changing with respect to time. We will need the same things each time- some given information and a formula that is appropriate to the circumstances

Let's consider the following problem:



RELATED RATE DEMO.

The situation.

An oil tanker anchored in a calm bay is leaking oil. As time goes on a circular oil slick is formed with the tanker at the center. Both the radius R and the area of the slick change with time.

Gulf of Mexico

If the radius of the oil slick is growing at a rate of 5 meters per hour, then how fast is the area of the oil slick growing at time = 3 hours?

Let's look at an animation of this problem. While you are watching the animation, think of what we need to solve a related rates problem.
http://astro.temple.edu/~dhill001/relatedrates/relatedrates.html

What steps do we need to take to solve this related rates problem?

$$a = \pi r^2$$

$$\text{find } \left.\frac{da}{dt}\right|_{t=3}$$

$$\frac{d}{dt} a = \frac{d}{dt} \pi r^2$$

$$\frac{da}{dt} = 2\pi r \frac{dr}{dt}$$

$$\frac{dr}{dt} = 5 \frac{m}{hr}$$

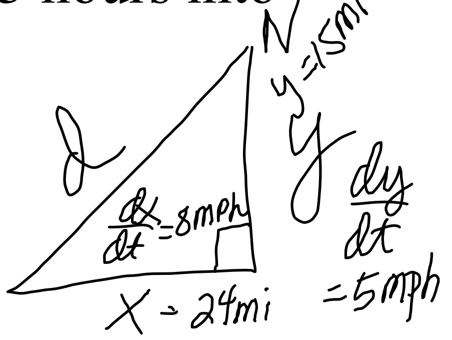
$$= (2\pi)(15m)\left(5 \frac{m}{hr}\right)$$

$$\frac{da}{dt} = 150\pi \frac{m^2}{hr}$$

Skippy and Binky are going to begin a hike at the same location and travel in perpendicular directions. Skippy travels due north at a rate of 5 miles per hour; Binky travels due west at a rate of 8 miles per hour. At what rate is the distance between them changing 3 hours into the hike?

Skippy

Binky



$$x^2+y^2=d^2$$

$$\frac{d}{dt}x^2+\frac{d}{dt}y^2=\frac{d}{dt}d^2$$

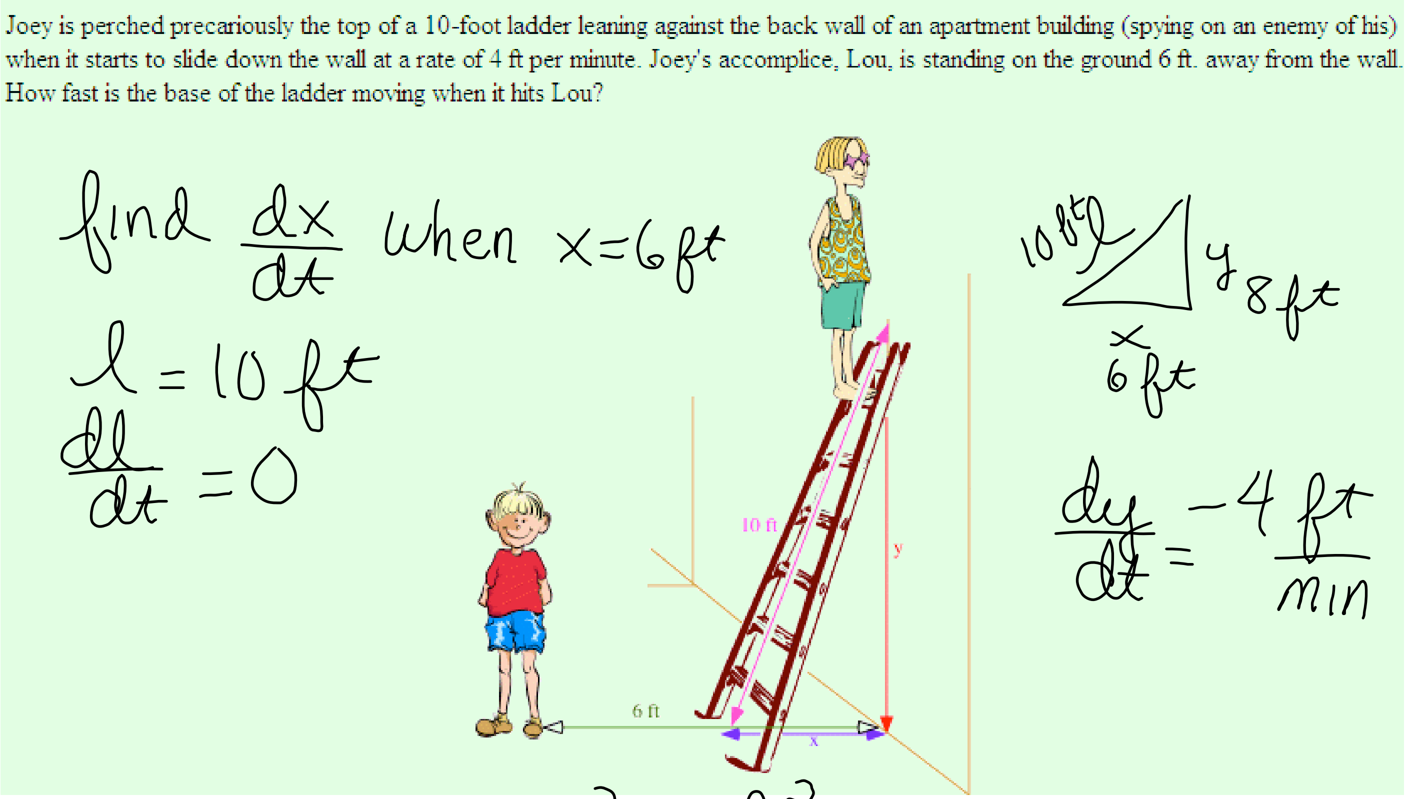
find $\left.\frac{dd}{dt}\right|_{t=3\,hr}$

$$\cancel{2}x\frac{dx}{dt}+\cancel{2}y\frac{dy}{dt}=\cancel{2}d\frac{dd}{dt}$$

$$(24\,mi)(8\,mph)+(15\,mi)(5\,mph)=\sqrt{801}\,\frac{dd}{dt}$$

$$\frac{267}{\sqrt{801}}=\frac{dd}{dt}$$

From: http://people.hofstra.edu/Faculty/Stefan_Waner/RealWorld/tutorials/frames4_4.html



Joey is perched precariously the top of a 10-foot ladder leaning against the back wall of an apartment building (spying on an enemy of his) when it starts to slide down the wall at a rate of 4 ft per minute. Joey's accomplice, Lou, is standing on the ground 6 ft. away from the wall. How fast is the base of the ladder moving when it hits Lou?

find $\frac{dx}{dt}$ when $x=6ft$

$l=10ft$

$\frac{dl}{dt}=0$

$\frac{dy}{dt}=-4\frac{ft}{min}$

$$x^2+y^2=l^2$$

$$\frac{d}{dt}x^2+\frac{d}{dt}y^2=\frac{d}{dt}l^2$$

$$\cancel{2}x\frac{dx}{dt}+\cancel{2}y\frac{dy}{dt}=\cancel{2}l\frac{dl}{dt}$$

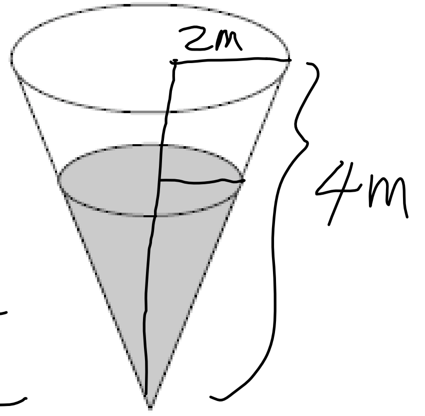
$$(6ft)\frac{dx}{dt}+(8ft)(-4\frac{ft}{min})=0$$

$$(6ft)\frac{dx}{dt}=32\frac{ft^2}{min}$$

$$\frac{dx}{dt}=\frac{32}{6}\frac{ft}{min}$$

From: http://chaoticgolf.com/

Example: A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep. The volume of a circular cone with radius r and height h is given by $V = \frac{1}{3}\pi r^2 h$.



$\frac{r}{h} = \frac{2}{4}$

$4r = 2h$

$r = \frac{1}{2}h$

Find $\frac{dh}{dt}\Big|_{h=3m}$

$\frac{dV}{dt} = \frac{2m^3}{min}$

$V = \frac{\pi}{3} r^2 h$

REWRITE

$V = \frac{\pi}{3}\left(\frac{h}{2}\right)^2 h$

$V = \frac{\pi}{12} h^3$

$\frac{d}{dt} V = \frac{d}{dt} \frac{\pi}{12} h^3$

$\frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}$

$2 \frac{m^3}{min} = \frac{\pi}{4}(3m)^2 \frac{dh}{dt}$

$\frac{dh}{dt}$ $\frac{8}{9\pi}$ $\frac{m}{min}$ =

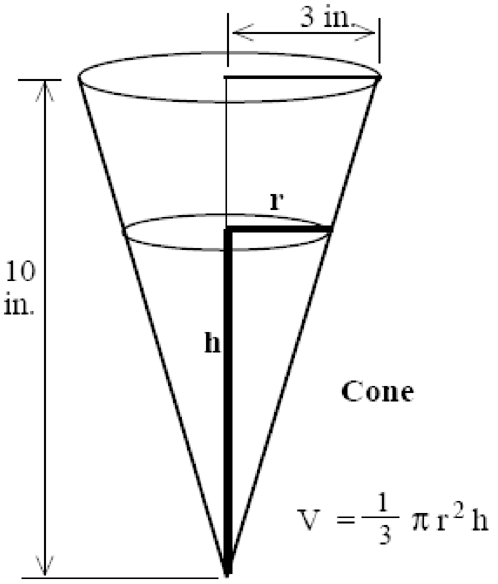
Homework pages 154-155 #15a, 19a, 22a, 23

Additional examples with solutions follow

Related Rates

http://scidiv.bcc.ctc.edu/dh/math124/Rel.Rate.Cone.pdf

Given: A cone that is 10 inches high and 6 inches wide (at the top).



Problem 1: What is the relationship between h (the height of the water) and r (the radius of the top circle of water)?

Problem 2: Write a formula for the volume of water
(a) as a function of h alone, and
(b) as a function or r alone.

Problem 3: Using the answers in 2(a) and 2(b), calculate $\frac{dV}{dt}$.

Problem 4: Translate the phrases
(a) "the height of the water is increasing 7 inches per minute,"
(b) "the radius if the surface is decreasing 2 inches per minute," and
(c) "water is pouring into the cone at a steady rate of 5 in^3/min"
into mathematical statements about derivatives.

Problem 5: When there is 6 inches of water in the cone, the height of the water is increasing at 7 inches per minuter. How fast is the volume of water in the cone increasing?

Problem 6: When there is 4 inches of water in the cone, the height of the water is increasing at 2 inches per minuter. How fast is the surface radius of water in the cone increasing?

Problem 7: $dV/dt = 5\ in^3/min$. How fast is the height of the water in the cone increasing when the water height is 4 inches?

Problem 8: $dV/dt = 5\ in^3/min$. How fast is the surface radius of the water in the cone increasing when the water height is 4 inches?

The radius r of a circle is increasing at a constant rate of 2 centimeters per minute. Find the rate of change of the area when (a) $r = 6$ centimeters and (b) $r = 24$ centimeters.

Solution

The variables r and A are related by $A = \pi r^2$. The rate of change of the radius is $\frac{dr}{dt} = 2$.

Equation: $A = \pi r^2$

Given Rate: $\frac{dr}{dt} = 2$

Find: $\frac{dA}{dt}$

$$\frac{d}{dt}[A] = \frac{d}{dt}[\pi r^2]$$

Differentiate with respect to t.

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

Chain Rule

a. When $r = 6$,

$$\frac{dA}{dt} = 2\pi(6)(2) = 24\pi \text{ cm}^2/\text{min}.$$

Substitute 6 for r and 2 for $\frac{dv}{dt}$.

b. When $r = 24$,

$$\frac{dA}{dt} = 2\pi(24)(2) = 96\pi \text{ cm}^2/\text{min}.$$

Substitute 24 for r and 2 for $\frac{dr}{dt}$.

A spherical balloon is inflated with gas at a rate of 500 cubic centimeters per minute. What is the rate of change of the radius when the radius is (a) 30 centimeters and (b) 60 centimeters?

Solution

Let V be the volume of the balloon and let r be its radius. Because the volume is increasing at a rate of 500 cubic centimeters per minute, you know that at time t the rate of change of the volume is $\frac{dV}{dt} = 500$. So, the problem can be stated as follows.

Given rate: $\frac{dV}{dt} = 500$ (constant rate)

Find: $\frac{dr}{dt}$ when $r = 30$ and $r = 60$

To find the rate of change of the radius, you must find an equation that relates the radius r to the volume V.

Equation: $V = \frac{4}{3}\pi r^3$ Volume of a sphere

Implicit differentiation with respect to t produces

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \qquad \text{Differentiate with respect to } t.$$

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\left(\frac{dV}{dt}\right). \qquad \text{Solve for } \frac{dr}{dt}.$$

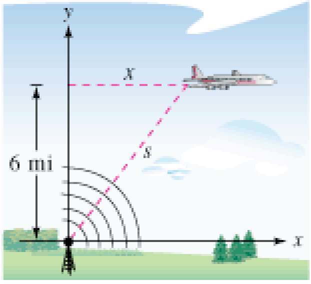
a. When $r = 30$, the rate of change of the radius is

$$\frac{dr}{dt} = \frac{1}{4\pi(30)^2}(500) = \frac{5}{36\pi} \text{ cm/min.}$$

b. When $r = 60$,

$$\frac{dr}{dt} = \frac{1}{4\pi(60)^2}(500) = \frac{5}{144\pi} \text{ cm/min.}$$

An airplane is flying at an altitude of 6 miles and passes directly over a radar antenna (see figure). When the plane is 10 miles away ($s = 10$), the radar detects that the distance s is changing at a rate of 240 miles per hour. What is the speed of the plane?



Solution

Let x be the horizontal distance from the radar antenna. Notice when $s = 10$, $x = \sqrt{10^2 - 36} = 8$.

Given rate: $\dfrac{ds}{dt} = 240$ when $s = 10$

Find: $\dfrac{dx}{dt}$ when $s = 10$ and $x = 8$

You can find the velocity of the plane as follows.

Equation: $x^2 + 6^2 = s^2$ — Pythagorean Theorem

$$2x\frac{dx}{dt} = 2s\frac{ds}{dt}$$ Differentiate with respect to t.

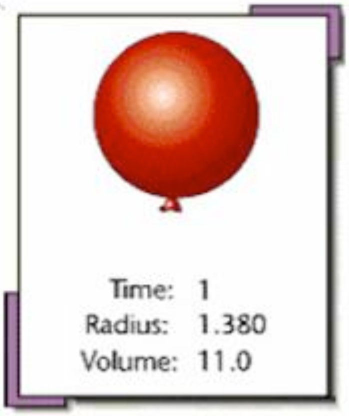
$$\frac{dx}{dt} = \frac{s}{x}\left(\frac{ds}{dt}\right)$$ Solve for $\frac{dx}{dt}$.

$$\frac{dx}{dt} = \frac{10}{8}(240)$$ Substitute for s, x, and $\frac{ds}{dt}$.

$$= 300 \text{ miles per hour}$$ Simplify.

Because the velocity is 300 miles per hour, the speed of the plane is 300 miles per hour.

2. Air is being pumped into a spherical balloon at a rate of 4.5 cubic inches per minute. Find the rate of change of the radius when the radius is 2 inches.



Equation(s) Needed: $V = \frac{4\pi}{3} r^3$

Given Rate(s): $\frac{dV}{dt} = 4.5 \ \frac{in^3}{min}$

Find: $\frac{dr}{dt}$ when $r = 2$ in.

$$V = \frac{4\pi}{3} r^3$$

$$\frac{d}{dt} V = \frac{d}{dt} \frac{4\pi}{3} r^3$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$4.5 \ \frac{in.^3}{min} = 4\pi (2 \ in.)^2 \frac{dr}{dt}$$

$$\frac{4.5}{16\pi} \ \frac{in.}{min} = \frac{dr}{dt}$$