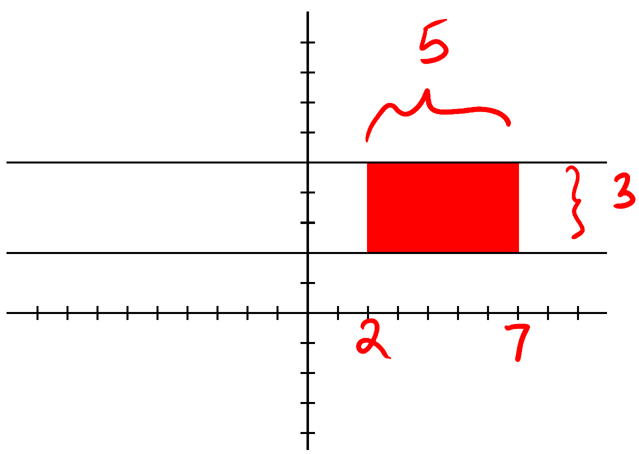
Area Between Two Curves

The simple case:

[I used http://cs.jsu.edu/mcis/faculty/leathrum/Mathlets/twocurves.html to generate these graphs.]

Let's find the area between the graphs of $y = 2$ and $y = 5$ on the interval $[2,7]$. Here's what it looks like:



Since it is just a rectangle, we can find the area by using

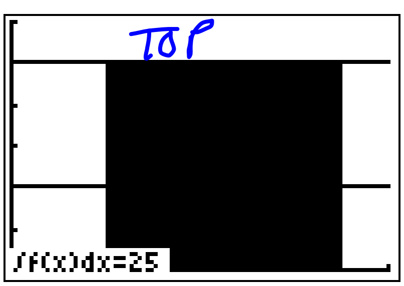
$A = l\,w$ where $l = 5$ and $w = 3$

$A = (5)(3)$

$A = 15\,\textit{square units}$

How could we do this with Calculus?

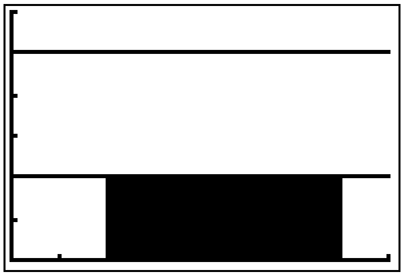
We could take all of the area from 2 to 7 of $y = 5$



$$\int_2^7 5\,dx$$

This is the "top" function.

and subtract away the area from 2 to 7 of $y = 2$



$$\int_2^7 2\,dx$$

This is the "bottom" function.

In other words:

$$\int_2^7 5\,dx - \int_2^7 2\,dx = \int_2^7 3\,dx = 3x\Big|_2^7$$

$$= 21 - 6$$

$$= 15$$

In textbook talk: [see page 447}

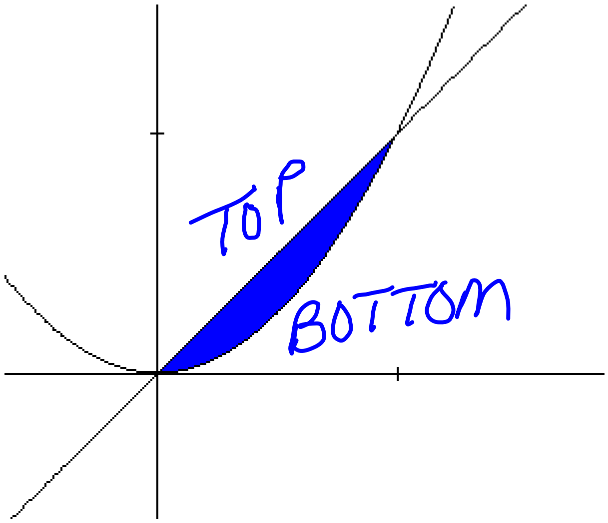
If f and g are continuous on $[a,b]$ AND $\overset{\text{Bottom}}{g(x)} \le \overset{\text{TOP}}{f(x)}$ for all x in $[a,b]$, then the area of the region bounded by the graphs of f and g and the vertical lines $\underset{\text{LOWER BOUND}}{x = a}$ and $\underset{\text{UPPER BOUND}}{x = b}$ is

$$A = \int_a^b \underbrace{[f(x) - g(x)]}_{\text{Top - Bottom}}\,dx$$

Find the area between $f(x) = x$ and $g(x) = x^2$ on [0, 1]

TOP BOTTOM

Here's what this looks like:



There is definitely a "top" function and a "bottom" function for this interval.

$$A = \int_0^1 \left[x - x^2\right] dx$$

$$A = \frac{x^2}{2} - \frac{x^3}{3} \Bigg|_0^1$$

$$A = \frac{1}{6}$$

Let's see page 452 #1 – 6

1. $\int_0^6 [0-(x^2-6x)]\,dx$ TOP: $g(x)$ BOTTOM: $f(x)$

2. $\int_{-2}^{2} [(2x+5)-(x^2+2x+1)]\,dx$ TOP: $g(x)$ Bottom: $f(x)$

3. $\int_0^3 [(-x^2+2x+3)-(x^2-4x+3)]\,dx$ TOP: $g(x)$ BOTTOM: $f(x)$

4. $\int_0^1 [x^2-x^3]\,dx$ TOP: $f(x)$ BOTTOM: $g(x)$

5. Be careful!

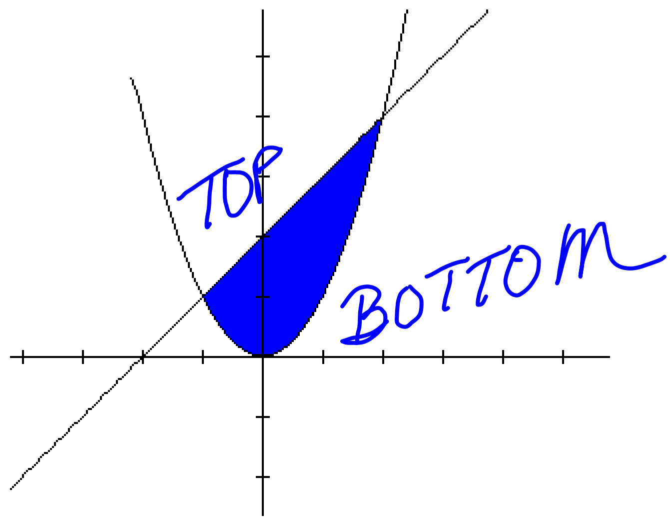
$$\int_{-1}^{0}[3(x^3-x)-0]\,dx + \int_0^1 [0-3(x^3-x)]\,dx$$

6. Be careful!

$$\int_0^1[(x-1)^3-(x-1)]\,dx + \int_1^2 [(x-1)-(x-1)^3]\,dx$$

Find the area bounded by $f(x) = x^2$ and $g(x) = x + 2$

1. Graph it!



2. Find points of intersection in order to find the bounds

$$x^2 = x + 2$$
$$x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0$$

$x = -1$
$x = 2$

3. Determine the "top" function and "bottom" function
4. Set up the integral and solve

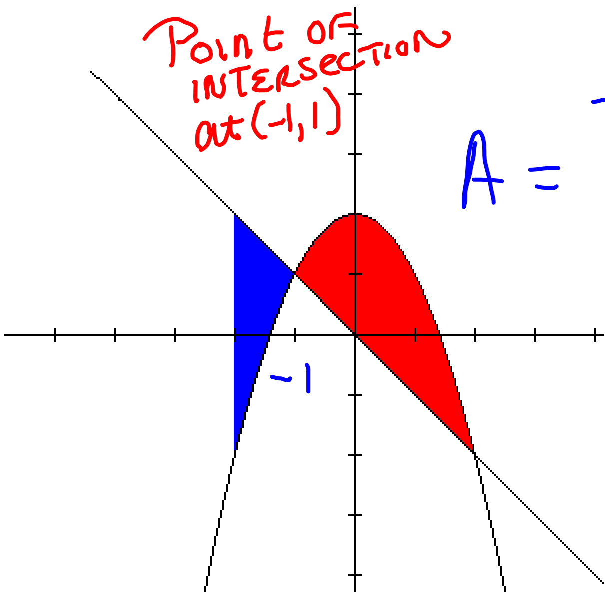
$$A = \int_{-1}^{2} \left[(x+2) - x^2\right]\, dx$$

$$A = 4.5 \textit{ sq units}$$

Another example:

Find the area between the graphs of $f(x) = 2 - x^2$ and $g(x) = -x$ on $[-2, 2]$.

$$2 - x^2 = -x$$
$$0 = x^2 - x - 2$$
$$0 = (x-2)(x+1)$$



$$A = \int_{-2}^{-1} [-x - (2 - x^2)]\,dx + \int_{-1}^{2} [(2 - x^2) - (-x)]\,dx = \frac{19}{3}$$

Hmmm! Notice how the "top" function switches at the intersection point. We need to find the intersection point.

From Mr. Zab's website: http://www.frapanthers.com/teachers/zab/APCalculusInaNutshell/ApplicationsofIntegrals.pdf

Finding the Area between Curves when the Functions Change Dominance

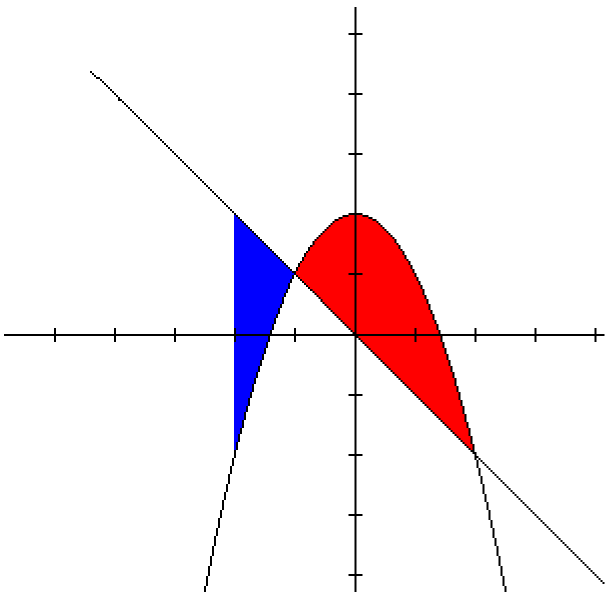
To find the area between the graphs of y = f(x) and y = g(x) on the interval [a, b] where
f and *g* change dominace over the interval.

1. partition [a, b] with the intersections of *f* and *g*,

2. write integrals which represent the area between the curves on the partitions and evaluate,

3. add the values of the integrals

What Mr. Zab means when he says “dominance” is the “top” function!



Intersection at $(-1,1)$

$$A = \int_{-2}^{-1} \left[(-x)-\left(2-x^2\right)\right] dx \;+\; \int_{-1}^{2} \left[\left(2-x^2\right)-(-x)\right] dx$$

See page 452 #13 and 14 [right – left]

$x = 4 - y^2$

$x = y - 2$

NOTICE:

x IN TERMS OF y

WE NEED "y" BOUNDS

$4 - y^2 = y - 2$

$0 = y^2 + y - 6$

$0 = (y+3)(y-2)$

$$\int_{-3}^{2} \left[(4-y^2)-(y-2)\right] dy$$

$$= \int_{-3}^{2} \left[6 - y - y^2\right] dy$$

$$= 6y - \frac{y^2}{2} - \frac{y^3}{3} \Big|_{-3}^{2} = \frac{22}{3} - \left(-\frac{27}{2}\right)$$

$$= \frac{125}{6}$$

Homework: Read 7.1 and do page 452 #17, 18, 20, 24, 25, 26- graph the region, set up the integral, integrate by hand, then check with your TI
Yes, draw the graph
Yes, do the integral by hand so that you get some good practice