

Other exponential functions:

We know that $\frac{d}{dx} e^x = e^x$ but what is $\frac{d}{dx} 2^x$?

No problem! We can use the **change of base formula** to rewrite: $2^x = e^{(\ln 2)x}$ [then we can use the e^u trick]

$$\text{So, } \frac{d}{dx} 2^x = \frac{d}{dx} (e^{(\ln 2)(x)})$$

$$\text{Let } \begin{aligned} u &= (\ln 2)(x) \\ \frac{du}{dx} &= \ln 2 \end{aligned}$$

$$\begin{aligned} &\frac{d}{dx} e^u \\ &= e^u \frac{du}{dx} \\ &= e^{(\ln 2)(x)} (\ln 2) \\ &= (\ln 2)(2^x) \end{aligned}$$

Does this always work? Yes it does! ☺

In general, $\frac{d}{dx} a^x = (\ln a)(a^x)$ Of course $a > 0$ because the ln function's domain is all positive numbers.

$$\frac{d}{dx} 711^x = (\ln 711)(711^x)$$

$(\ln a)(a^x)$

$$a = 711$$

$$\frac{d}{dx} 13^x = (\ln 13)(13^x)$$

$$a=13$$

In general, $\frac{d}{dx} a^u = (\ln a)(a^u) \frac{du}{dx}$

$a > 0$, u is a differentiable function of x

$$\frac{d}{dx} 7^{11x}$$

$$a=7, u=11x$$

$$\frac{du}{dx} = 11$$

$$= (\ln 7)(7^{11x})(11)$$
$$= (\ln a)(a^u) \frac{du}{dx}$$

$$\frac{d}{dx} 6^x$$
$$= (\ln 6)(6^x)$$

$$a=6$$

$$\frac{d}{dx} a^x = (\ln a)(a^x)$$

$$\frac{d}{dx} 6^{2x}$$
$$= (\ln 6)(6^{2x})(2)$$

$$a=6 \quad \left\{ \begin{array}{l} \frac{d}{dx} a^u \\ u=2x \\ \frac{du}{dx} = 2 \end{array} \right. = (\ln a)(a^u) \frac{du}{dx}$$

Integrating with a^x , a^u

$$\text{In general, } \int a^x dx = \frac{1}{\ln a} (a^x) + C$$

$$a \in \mathbb{R} \\ a > 0$$

$$\int 711^x dx = \frac{1}{\ln(711)} (711^x) + C$$

$$a = 711$$

$$\begin{aligned} & \int (7^x + 11^x) dx \\ &= \int 7^x dx + \int 11^x dx \\ &= \frac{1}{(\ln 7)} (7^x) + \frac{1}{(\ln 11)} (11^x) + C \end{aligned}$$

You try:

$$\int 37^x dx$$

$$a = 37$$

$$= \frac{1}{\ln 37} (37^x) + C$$

Inverse Trig Functions

We need to know two for Calc AB [Calc BC needs more]

$$\begin{aligned}\frac{d}{dx}(\arctan x) \\ &= \frac{d}{dx}(\tan^{-1} x) \\ &= \frac{1}{1+x^2}\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}(\arcsin x) \\ &= \frac{d}{dx}(\sin^{-1} x) \\ &= \frac{1}{\sqrt{1-x^2}}\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}(\arccos x) \\ &= \frac{d}{dx}(\cos^{-1} x) \\ &= \frac{-1}{\sqrt{1-x^2}}\end{aligned}$$

PLEASE MEMORIZE THESE!!!!!!!

$$\int \frac{1}{1+x^2} dx = \arctan x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$$

$$\int \frac{-dx}{\sqrt{1-x^2}} = \arccos x + C \quad \mathbf{OR}$$

$$\int \frac{-dx}{\sqrt{1-x^2}} = - \int \frac{dx}{\sqrt{1-x^2}} = -\arcsin x + C$$

Beware

*$\int \frac{2x}{1+x^2} dx$
this is
A U-SUB
PROBLEM*

The Chain Rule still applies!!! [u is a differentiable function of x]

$$\frac{d}{dx} \arctan u = \frac{1}{1+u^2} \cdot \frac{du}{dx}$$

Example: $\frac{d}{dx} \arctan(3x^2)$

$$u = 3x^2$$

$$\frac{du}{dx} = 6x$$

OUR HANDY FORMULA

$$= \frac{1}{1+u^2} \cdot \frac{du}{dx}$$

$$= \frac{1}{1+9x^4} (6x)$$

$$= \frac{6x}{1+9x^4}$$

$$\frac{d}{dx} \arcsin u = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$$

Example: $\frac{d}{dx} \arcsin(3x^2)$

$$u = 3x^2$$

$$\frac{du}{dx} = 6x$$

OUR HANDY FORMULA

$$= \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$$

$$= \frac{1}{\sqrt{1-9x^4}} (6x)$$

$$= \frac{6x}{\sqrt{1-9x^4}}$$

Integrating with inverse trigonometric functions

$$\int \frac{1}{1+x^2} dx = \arctan x + C$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a} \arctan \frac{u}{a} + C$$

Don't forget

a is a NUMBER
u is a DIFF. FUNCTION OF X

Don't get TRICKED

$$\int \frac{10}{1+x^2} dx = 10 \int \frac{1}{1+x^2} dx = 10 \arctan x + C$$

Examples:

$$a = 2$$

$$\int \frac{3 dx}{4 + 9x^2}$$

$$\text{Let } u = 3x$$

$$= \int \frac{du}{a^2 + u^2}$$

*SUM OF 2 SQUARES
in DENOM.*

$$du = 3dx$$

*and U-SUB DOESN'T
WORK!*

$$= \frac{1}{a} \arctan \frac{u}{a} + C$$

$$= \frac{1}{2} \arctan \left(\frac{3x}{2} \right) + C$$

Let's check:

$$\frac{d}{dx} \left[\frac{1}{2} \arctan \left(\frac{3x}{2} \right) + C \right]$$

$$u = \frac{3x}{2}$$

$$= \frac{1}{2} \frac{d}{dx} \left[\arctan \left(\frac{3x}{2} \right) + C \right]$$

$$\frac{du}{dx} = \frac{3}{2}$$

$$= \frac{1}{2} \left(\frac{1}{1 + \left(\frac{3x}{2} \right)^2} \right) \left(\frac{3}{2} \right) + 0$$

$$\begin{aligned}
&= \frac{3}{4} \left(\frac{1}{1 + \frac{9x^2}{4}} \right) \\
&= \frac{3}{4} \left(\frac{1}{\frac{4 + 9x^2}{4}} \right) \\
&= \frac{3}{4 + 9x^2}
\end{aligned}$$

Now simplify!

Likewise, $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$

What would $\int \frac{3dx}{\sqrt{1-x^2}}$ equal?

Rewrite as: $3 \int \frac{dx}{\sqrt{1-x^2}}$
 $= 3 \arcsin x + C$

Also, $\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin \frac{u}{a} + C$

*a is A NUMBER
u is A DIFF function
of x*

Example: $\int \frac{dx}{\sqrt{9-x^2}}$

In this case, $a^2 = 3^2$ so $a = 3$; $u = x$ so $du = dx$

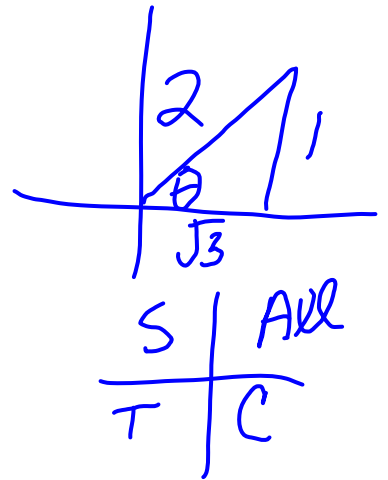
Hence, $\int \frac{dx}{\sqrt{9-x^2}} = \arcsin \frac{x}{3} + C$

Here is an example of a definite integral with arcsin:

$$\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}} = \arcsin x \Big|_0^{\frac{1}{2}}$$

$$= \arcsin\left(\frac{1}{2}\right) - \arcsin(0)$$

$$= \frac{\pi}{6}$$



For a really good summary of Basic Integration Rules, see page 383. [We don't need arcsec(x)]

Homework: page 366 #37, 39, 49; page 367 #61, 67, 69; page 378 #41, 45, 59; page 385 #1, 3