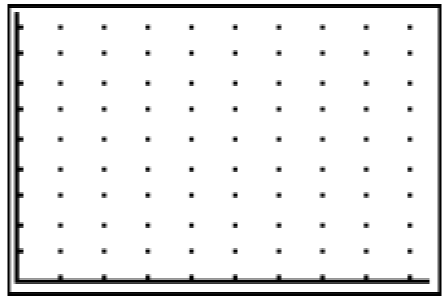
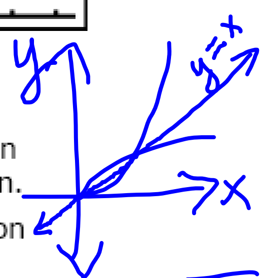
DISCOVERING SPECIAL RELATIONSHIPS
BETWEEN DERIVATIVES OF INVERSE FUNCTIONS.



Set the window to [0,9,4,1, 0, 9.4,1,1].
Graph $y_1 = f(x) = \sqrt{x}$. Turn the grid on using 2nd Format. Record your window at the right.

To view the inverse of this function return to the home screen and enter the command DrawInv Y1 (2nd DRAW, 8. DrawInv and then Vars, Y-VARS, 1. Function, 1. Y1). If the original function is f(x), then the inverse function is $f^{-1}(x)$.



Record this function in the same window. Clearly define this new inverse function with its restricted domain and enter the equation in Y2 with the restricted domain.
$f^{-1}(x) = =$ x^2 $(x \geq 0)$ Confirm that you have written the correct equation using the graphs.

Over what line has f(x) been reflected to create the inverse function? $y = x$

$y = \sqrt{x}$
$x = \sqrt{y}$
$x^2 = y$

If (4, f(4)) or (4,2) is a point on the original function, what is the name of the point on the inverse function which corresponds to this? (2, 4) Mark both of these on the sketch.

$y = \sqrt{x}$ $y' = \frac{1}{2\sqrt{x}}$

To find the slope at both of these points use 2nd CALC, 6. dy/dx, select the correct graph, and enter the number where you want the slope.

Find the slope of the original function at (4, f(4)) or (4,2). $f'(4) = \frac{1}{4}$

$f^{-1} = x^2$

Find the slope of the inverse function at the corresponding point. 4

$\frac{d}{dx}[f^{-1}]$

If (9,f(9)) or (9,3) is a point on the original function, what is the name of the point on the inverse function which corresponds to this? (3, 9) Mark both of these on the sketch.

Find the slope of the original function at (9, f(9)) or (9,3). $f'(9) = \frac{1}{6}$

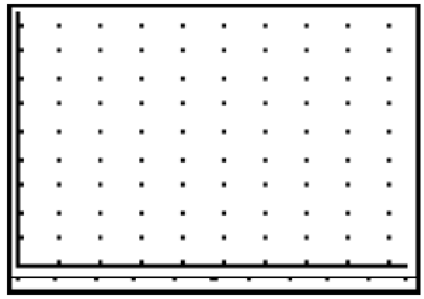
$= 2x$

Find the slope of the inverse function at the corresponding point. 6

What is the relationship between the slope of these two functions at their corresponding points?

RECIPROCALS

2. Let's try another function: Graph $g(x)=2x-2$ in the same window. Record a sketch of this equation in the figure at the right.



$y=2x-2$

$x=2y-2$

$\frac{x+2}{2}=y$

$.5x+1=y$

Graph the inverse function $g^{-1}(x)$ the same way you did in exercise 1.

What is the equation of the inverse function?

$g^{-1}(x)=\frac{1}{2}x+1$ Enter this equation in y2. Confirm that you have written the correct equation using the graphs.

Locate two corresponding points on the graph of g: (2,g(2)) or (2,2) and (3,g(3)) or (6,4). Mark these on the sketch. Locate the two corresponding points on the graph of g^{-1} .

What is the derivative of g(x)? $g'(x)=2$

What is the derivative of $g^{-1}(x)$? $\frac{d}{dx}[g^{-1}(x)]=\frac{1}{2}$

What is the slope at each of the functions at the selected points?

The activity above is the property of http://jamesrahn.com

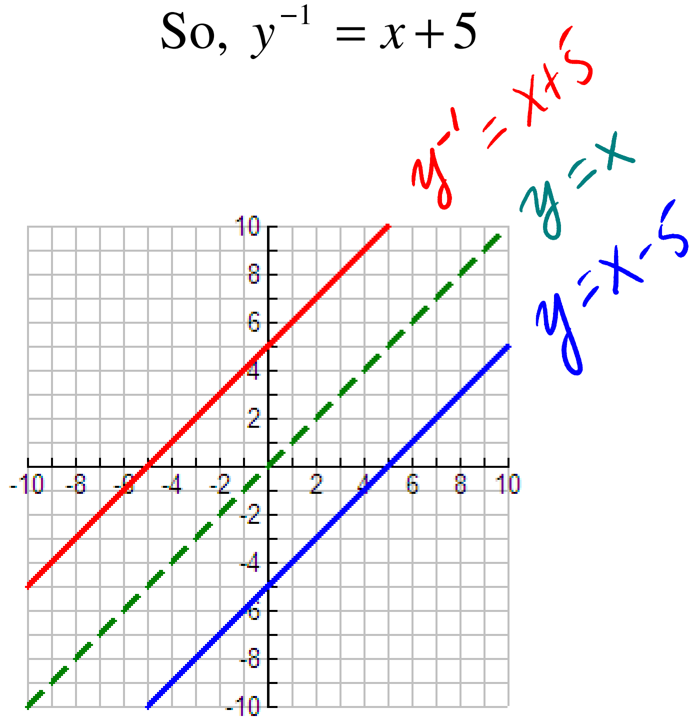
Inverse Functions

If $y = x - 5$, then what is y^{-1}?

$x = y - 5$

$x + 5 = y$

So, $y^{-1} = x + 5$



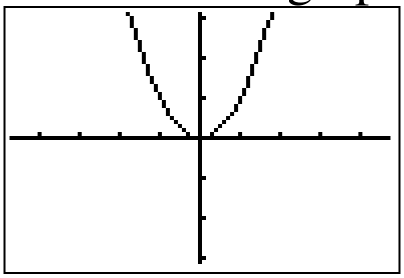
$f^{-1}(x)$ is a reflection across the line $y = x$.

If the graph of f contains the point (a, b), then the graph of f^{-1} contains the point (b, a)

x	$f(x) = x - 5$
0	- 5
5	0

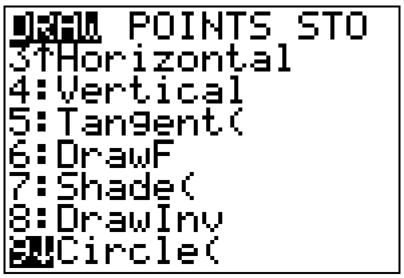
x	$f^{-1}(x) = x + 5$
-5	0
0	5

Your TI will graph any inverse for you.

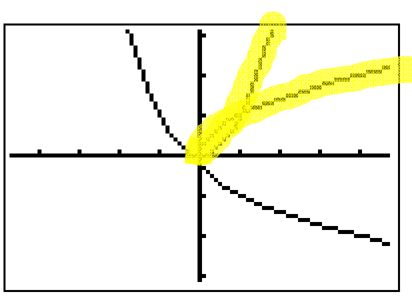


Let $y_1 = x^2$

Go to the DRAW Menu and choose option 8 "DrawInv"



We'll get this result:



If $f(x) = x^2$, then $f^{-1}(x) = \sqrt{x}$ $\quad x \geq 0$

$$f'(x) = 2x \text{ and } \frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{2\sqrt{x}}$$

x	$f(x)$ $y=x^2$	$f'(x)$ $y'=2x$	☺	x	$f^{-1}(x)$ $y^{-1}=\sqrt{x}$	$\frac{d}{dx}\left(f^{-1}(x)\right)$
1	1	2	☼	1	1	$\frac{1}{2}$
2	4	4	♥	4	2	$\frac{1}{4}$
3	9	6	♫	9	3	$\frac{1}{6}$
4	16	8	⁂	16	4	$\frac{1}{8}$

Compare the derivative columns. Hmm!

Let $f(x)=x^3$, then $f^{-1}(x)=x^{\frac{1}{3}}$

$f'(x)=3x^2$ and $\frac{d}{dx}\left(f^{-1}(x)\right)=\frac{1}{3}x^{\frac{-2}{3}}$ $= \frac{1}{3\sqrt[3]{x^2}}$

Let's make a chart to compare again.

x	$f(x)$	$f'(x)$	☯	x	$f^{-1}(x)$	$\frac{d}{dx}\left(f^{-1}(x)\right)$
1	1	3	✵	1	1	$\frac{1}{3}$
2	8	12	ω	8	2	$\frac{1}{12}$
3	27	27	ϟ	27	3	$\frac{1}{27}$

Theorem

Let f be a function that is differentiable on an interval I. If f has an inverse ***function*** g, then g is differentiable at any x for which $f'(g(x)) \neq 0$ AND

$$g'(x) = \frac{1}{f'(g(x))}$$

The theorem in action:

$f(x) = x^3, \; f'(x) = 3x^2$

The point $(5,125)$ is on the graph of $f(x)$ and $f'(5) = 75$. From the theorem, we know that the point $(125,5)$ is on the graph of $f^{-1}(x)$ and if we let $g(x) = f^{-1}(x)$, then we also know that $g'(125) = \frac{1}{75} = \frac{1}{f'(5)}$ $\quad g(125) = 5$

If $f(2) = 5, \; f'(2) = -3$, then what do we know about $f^{-1}(x) = g(x)$ and $g'(x)$?

$g(5) = 2 \qquad g'(5) = \frac{1}{f'(2)} = \frac{1}{-3}$

If $f(3) = 0, \; f'(3) = -6$, then

$g(0) = 3 \qquad g'(0) = \frac{1}{f'(3)} = \frac{1}{-6}$

If $f(12) = -1, \; f'(12) = 6$, then

$g(-1) = 12 \qquad g'(-1) = \frac{1}{f'(12)} = \frac{1}{6}$

My Inverse Function Worksheet ______________________________

Let $g(x) = f^{-1}(x)$ and use the given information to find the appropriate value of $g'(x)$

1. Given: $f(x) = x^4$, $f(2) = 16$

$f'(x) = 4x^3$

$g(16) = 2 \quad g'(16) = \frac{1}{f'(2)} = \frac{1}{32}$

2. $f(x) = 3 - 4x$, $f(1) = -1$

$g(-1) = 1 \quad g'(-1) = \frac{1}{f'(1)} = \frac{1}{-4}$

3. $f(x) = \sqrt{x-4}$, $f(5) = 1$

$g(1) = 5 \quad g'(1) = \frac{1}{f'(5)} = 2$

$f'(x) = \frac{1}{2\sqrt{x-4}}$

$f'(5) = \frac{1}{2}$

4. $f(x) = \sin x$, $f\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$g\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3} \quad g'\left(\frac{\sqrt{3}}{2}\right) = \frac{1}{f'\left(\frac{\pi}{3}\right)} = 2$

5. $f(x) = x^2$, $f(6) = 36$

$g(36) = 6 \quad g'(36) = \frac{1}{f'(6)} = \frac{1}{12}$

6. $f(x) = \ln(x^2 + 1)$, $f(2) = \ln 5$

$g(\ln 5) = 2 \quad g'(\ln 5) = \frac{1}{f'(2)} = \frac{5}{4}$

$f'(x) = \frac{1}{x^2+1}(2x)$

$f'(2) = \frac{4}{5}$

7. Let $g = f^{-1}$ and $f(x) = 7x^2 - 5x + 3$. Find $g'(21)$ [$(x>0)$]

$21 = 7x^2 - 5x + 3$ at $x = 2$

$g(21) = 2$ $g'(21) = \frac{1}{f'(2)} = \frac{1}{23}$

8. Let $g = f^{-1}$ and $f(x) = x^2 + 3x + 1$. Find $g'(29)$ [$x>0$]

$29 = x^2 + 3x + 1$ at $x = 4$

$g(29) = 4$ $g'(29) = \frac{1}{f'(4)} = \frac{1}{11}$

9. Let $g = f^{-1}$ and $f(x) = x^3 + 2x^2 - 10$. Find $g'(6)$ [$x > 0$]

$6 = x^3 + 2x^2 - 10$ at $x = 2$

$g(6) = 2$ $g'(6) = \frac{1}{f'(2)} = \frac{1}{20}$

10. Let $g = f^{-1}$ and $f(x) = 6x^2 + 4x - 2$. Find $g'(8)$ [$x > 0$]

$8 = 6x^2 + 4x - 2$ at $x = 1$

$g(8) = 1$ $g'(8) = \frac{1}{f'(1)} = \frac{1}{16}$

11. Let $g = f^{-1}$ and $f(x) = 5x^3 - 10x^2$. Find $g'(45)$ [$x > 0$]

$45 = 5x^3 - 10x^2$ at $x = 3$

$g(45) = 3$ $g'(45) = \frac{1}{f'(3)} = \frac{1}{75}$

12. Let $g = f^{-1}$ and $f(x) = x^4 - 5x$. Find $g'(6)$ [$x > 0$]

$6 = x^4 - 5x$ at $x = 2$

$g(6) = 2$ $g'(6) = \frac{1}{f'(2)} = \frac{1}{27}$