Integrating with $\ln x$

$\int \frac{1}{x}\, dx = \ln|x| + C$ **[Beware of domain issues]**
IF NOT SPECIFIED THAT $x > 0$

If u is a differentiable function of x, then $\int \frac{1}{u}\, du = \ln|u| + C$

Examples

$\int \frac{3}{x} dx$

We can think of this as $3\int \frac{1}{x} dx$ [our friend, the constant multiple rule]

$\int \frac{3}{x} dx =$

$3\int \frac{1}{x} dx =$

$3\ln|x| + C$

Another example:

$\int \frac{1}{3x}\, dx$

$= \frac{1}{3}\int \frac{1}{x} dx$

$= \frac{1}{3}\ln|x| + C$

Because

$\frac{1}{3x} = \left(\frac{1}{3}\right)\left(\frac{1}{x}\right)$

Slightly harder:

$$\int \frac{3}{3x+2}\,dx$$

Remember: the division bar is a grouping symbol.

We can use the u-sub here!

$$\int \frac{3}{3x+2}\,dx \qquad \text{Let } \begin{array}{l} u = 3x+2 \\ du = 3\,dx \end{array}$$

So this can be rewritten as:

$$\int \frac{du}{u} \text{ which is equivalent to } \int \frac{1}{u}\,du$$

$$= \ln|u| + C$$

$$= \ln|3x+2| + C$$

Let's change just one thing:

$\int \frac{dx}{3x+2}$

Let $u = 3x+2$

Then $du = 3\,dx$

And $\frac{1}{3}du = dx$

So this can be rewritten as:

$$\int \frac{dx}{3x+2}$$

$$= \frac{1}{3}\int \frac{1}{u}\, du$$

$$= \frac{1}{3}\ln|u| + C$$

$$= \frac{1}{3}\ln|3x+2| + C$$

Slightly harder:

$\int \frac{4x}{2x^2+3}\, dx$ ♪ ☆ $2x^2+3>0$ for all x

$u = 2x^2+3$

$du = 4x\, dx$

$$= \int \frac{du}{u}$$

$$= \ln|u| + C$$

$$= \ln|2x^2+3| + C \text{ or } \ln(2x^2+3)^{☆} + C$$

see above

Don't get tricked into thinking that every integral with a division involves $\ln x$ or $\ln u$!

$$\int \frac{3x^2}{(x^3+4)^4}\,dx$$

$u = x^3+4$

$du = 3x^2\,dx$

$= \int u^{-4}\,du$

$= \frac{u^{-3}}{-3} + C$

$= -\frac{1}{3}(x^3+4)^{-3} + C$

NOT A $\ln u$ PROBLEM

Sometimes it is difficult to see how $\ln u$ is involved unless we simplify first:

$$\int \frac{x^3 - 3x^2 + 5}{x-3}\,dx$$

The degree of the numerator is greater than the degree of the denominator, so try dividing first!

$$\begin{array}{r|l} & x^2 + \frac{5}{x-3} \\ x-3 & x^3 - 3x^2 + 5 \\ & x^3 - 3x^2 \\ \hline & \quad\quad 5 \end{array}$$

So, $\int \frac{x^3 - 3x^2 + 5}{x-3}\, dx$ can be rewritten as

$$\int \left(x^2 + \frac{5}{x-3} \right) dx$$

$$= \int x^2\, dx + 5\int \frac{1}{x-3}\, dx$$

$$= \frac{x^3}{3} + 5\ln|x-3| + C$$

$u = x - 3$
$du = dx$
think $5\int \frac{du}{u}$

See Exploration on page 332

(1) $\int \frac{2}{x} dx$

$$= 2\int \frac{1}{x} dx$$

(2) $\int \frac{1}{4x-1} dx$

$u = 4x - 1$
$du = 4dx$
$\frac{1}{4} du = dx$

$$= \frac{1}{4} \int \frac{1}{u}\, du$$

(3) $\int \frac{x}{x^2+2}\,dx$

$u = x^2+2$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$= \frac{1}{2}\int \frac{1}{u}\,du$

(4) $\int \frac{3x^2+1}{x^3+x}\,dx$

$u = x^3+x$

$du = (3x^2+1)\,dx$

$= \int \frac{1}{u}\,du$

(5) $\int \frac{x+1}{x^2+2x}\,dx$

$u = x^2+2x$

$du = (2x+2)\,dx$

$\frac{1}{2}du = (x+1)\,dx$

$= \frac{1}{2}\int \frac{1}{u}\,du$

(6) $\int \frac{1}{3x+2}\,dx$

done :)

(7) $\int \frac{x^2+x+1}{x^2+1}\,dx$

$$
\begin{array}{r}
1 + \frac{x}{x^2+1} \\
x^2+1 \,\big)\, \overline{x^2+x+1} \\
\underline{x^2 \quad\; +1} \\
x
\end{array}
$$

$= \int \left[1 + \frac{x}{x^2+1}\right] dx$

$= \int 1\,dx + \int \frac{x}{x^2+1}\,dx$

$u = x^2+1$

$du = 2x\,dx$

$\frac{1}{2}\,du = x\,dx$

$\frac{1}{2}\int \frac{du}{u}$

(8) $\int \frac{2x}{(x+1)^2}\,dx$

$= \int \frac{2x}{x^2+2x+1}\,dx$ see page 334

ACK! NOT HELPFUL!

BUT HERE'S YOUR HINT: let $u = x+1$

then $u-1 = x$

$du = dx$

$\int \frac{2(u-1)}{u^2}\,du = \int \frac{2u-2}{u^2}\,du$

$= \int \frac{2u}{u^2}\,du - \int \frac{2}{u^2}\,du$

Common $\ln x$ integration problems:

$\int \frac{dx}{x \ln x}$ Can be rewritten as:

$\frac{3+2}{8} = \frac{3}{8} + \frac{2}{8}$

$= \int \left(\frac{1}{x}\right)\left(\frac{1}{\ln x}\right) dx$ Let $u = \ln x$, $du = \frac{1}{x}\,dx$

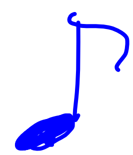
$= \int \frac{1}{u}\,du = \ln|u| + C$

$\frac{1}{3} \cdot \frac{1}{5} = \frac{1}{15}$

$\cancel{\frac{1}{3+5} = \frac{1}{3} + \frac{1}{5}}$

$= \ln|\ln x| + C$

$\int \frac{(\ln x)^2}{x}\,dx$ Can be rewritten as:



$= \int \left(\frac{1}{x}\right)(\ln x)^2\,dx$ Let $u = \ln x$, $du = \frac{1}{x}\,dx$

$(\ln x)^2 \neq \ln x^2$

$\int u^2\,du$

$= \frac{u^3}{3} + C$

$= \frac{(\ln^3 x)}{3} + C$ or $\frac{(\ln x)^3}{3} + C$

It pays to remember our natural log rules:

$$\int \frac{dx}{x\ln(x^2)}$$

Can be simplified and rewritten as

$\ln x^2 = 2\ln x$

$$= \int \frac{dx}{2x\ln x}$$

which now looks familiar!

$$= \frac{1}{2}\int \left(\frac{1}{x}\right)\left(\frac{1}{\ln x}\right) dx$$

$$= \frac{1}{2}\ln\left|\ln x\right| + C$$

And now we can find some trig integrals that we were unable to do until now:

$$\int \tan x\, dx$$

Let $u = \cos x$

$du = -\sin x\, dx$

$-du = \sin x\, dx$

$$= \int \frac{\sin x}{\cos x} dx$$

$$= -\int \frac{du}{u}$$

$$= -\ln|u| + C$$

$$= -\ln|\cos x| + C$$

If you are "fancy" $= \ln|\cos x|^{-1} + C = \ln|\sec x| + C$

You try: $\int \cot x\, dx$

$u = \sin x$

$du = \cos x\, dx$

$$= \int \frac{\cos x}{\sin x} dx$$

$$= \int \frac{1}{u}\, du = \ln|u| + C$$

$$= \ln|\sin x| + C$$

$$\int \tan(7x)\,dx$$

$$= \int \frac{\sin(7x)}{\cos(7x)}\,dx$$

$$= -\frac{1}{7}\int \frac{1}{u}\,du$$

$u = \cos(7x)$

$du = -7\sin(7x)\,dx$

$-\frac{1}{7}\,du = \sin(7x)\,dx$

$u = \tan x + 5$

$du = \sec^2 x\,dx$

$$\int \frac{\sec^2 x}{\tan x + 5}\,dx$$

$$= \int \frac{1}{u}\,du$$

A really weird integral!

$\int \sec x\,dx$ We can multiply by a clever form of one

$$= \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x}\, dx$$

Let $u =$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx$$

Definite Integrals [same as last chapter's]

$$\int_1^2 \frac{1 - \cos\theta}{\theta - \sin\theta}\, d\theta$$

Let
$u =$
$du =$
$u(1) =$
$u(2) =$

You try:

$$\int_1^e \frac{(1+\ln x)^2}{x}\,dx$$

Let's not forget the Second FTC!

Let $F(x)=\int_1^x \frac{1}{t}\,dt$ Find $F'(x)$

Let $F(x)=\int_1^{3x} \frac{1}{t}\,dt$ Find $F'(x)$

Let $F(x)=\int_{\frac{\pi}{4}}^{x} \cot t\,dt$ Find $F'(x)$

Our first separable differential equation!

Solve the following separable diff eq:

SOLVE for y

$$\frac{dy}{dx} = \frac{3}{2-x}$$ given: (1,0)is a point on the curve

$$dy = \frac{3}{2-x}\,dx$$ Separate!

$$\int dy = \int \frac{3}{2-x}\,dx$$ Integrate!

$$y + C_1 = -3\ln|2-x| + C_2$$ Combine the constants.

$$y = -3\ln|2-x| + C$$ Solve for C using the given point

Let $x = 1,\ y = 0$

$$0 = -3\ln|2-1| + C$$ Hence $C = 0$

$$y = -3\ln|2-x|$$ Solve for y

You try:

$$\frac{dy}{dx} = \frac{2x}{x^2+1}$$

Homework: pages 338 and 339 #2, 4, 6, 7, 10, 12, 20, 30, 34, 35, 36, ~~38~~, ~~40~~, ~~48~~, ~~50~~, 62, 64

NOT these