Integrating with $\ln x$

$\int \frac{1}{x}\, dx = \ln|x| + C$ **[Beware of domain issues]**

If u is a differentiable function of x, then $\int \frac{1}{u}\, du = \ln|u| + C$

Examples

$\int \frac{3}{x} dx$

We can think of this as $3\int \frac{1}{x} dx$ [our friend, the constant multiple rule]

$\int \frac{3}{x} dx =$

$3\int \frac{1}{x} dx =$

$3\ln|x| + C$

Another example:

$\int \frac{1}{3x}\, dx$

$= \frac{1}{3}\int \frac{1}{x} dx$

$= \frac{1}{3}\ln|x| + C$

Slightly harder:

$$\int \frac{3}{3x+2}\,dx$$

Remember: the division bar is a grouping symbol.

We can use the u-sub here!

$$\int \frac{3}{3x+2}\,dx \qquad \text{Let } \begin{array}{l} u = 3x+2 \\ du = 3\,dx \end{array}$$

So this can be rewritten as:

$\int \frac{du}{u}$ which is equivalent to $\int \frac{1}{u}\,du$

$= \ln|u| + C$

$= \ln|3x+2| + C$

Let's change just one thing:

$\int \frac{dx}{3x+2}$

Let $u = 3x + 2$

Then $du = 3\,dx$

And $\frac{1}{3}du = dx$

So this can be rewritten as:

$$\int \frac{dx}{3x+2}$$

$$= \frac{1}{3}\int \frac{1}{u}\ du$$

$$= \frac{1}{3}\ln|u| + C$$

$$= \frac{1}{3}\ \ln|3x+2| + C$$

Slightly harder:

$\int \frac{4x}{2x^2+3}\ dx$ ♪ $2x^2 + 3 > 0$ for all x

Don't get tricked into thinking that every integral with a division involves $\ln x$ or $\ln u$!

$$\int \frac{3x^2}{\left(x^3+4\right)^4}\,dx$$

Sometimes it is difficult to see how $\ln u$ is involved unless we simplify first:

$$\int \frac{x^3-3x^2+5}{x-3}\,dx$$

The degree of the numerator is greater than the degree of the denominator, so try dividing first!

$$x-3\;\overline{)\,x^3-3x^2+5}$$

So, $\int \frac{x^3 - 3x^2 + 5}{x-3}\, dx$ can be rewritten as

$$\int \left(x^2 + \frac{5}{x-3} \right) dx$$

$$= \int x^2\, dx + 5\int \frac{1}{x-3}\, dx$$

$$= \frac{x^3}{3} + 5\ln|x-3| + C$$

See Exploration on page 332

(1) $\int \frac{2}{x} dx$

(2) $\int \frac{1}{4x-1} dx$

(3) $$\int \frac{x}{x^2+2}\,dx$$

(4) $$\int \frac{3x^2+1}{x^3+x}\,dx$$

(5) $$\int \frac{x+1}{x^2+2x}\,dx$$

(6) $\int \frac{1}{3x+2}\,dx$

(7) $\int \frac{x^2+x+1}{x^2+1}\,dx$

(8) $\int \frac{2x}{(x+1)^2}\,dx$

Common $\ln x$ integration problems:

$$\int \frac{dx}{x \ln x}$$

Can be rewritten as:

$$= \int \left(\frac{1}{x}\right)\left(\frac{1}{\ln x}\right) dx$$

Let $u = \ln x$, $du = \frac{1}{x} dx$

$$= \int \frac{1}{u} du$$

$$= \ln\left|\ln x\right| + C$$

$$\int \frac{(\ln x)^2}{x} dx$$

Can be rewritten as:

$$= \int \left(\frac{1}{x}\right)(\ln x)^2 \ dx$$

Let $u = \ln x$, $du = \frac{1}{x} dx$

$$\int u^2 \, du$$

$$= \frac{u^3}{3} + C$$

$$= \frac{\left(\ln^3 x\right)}{3} + C$$

It pays to remember our natural log rules:

$$\int \frac{dx}{x\ln(x^2)}$$ Can be simplified and rewritten as

$$= \int \frac{dx}{2x\ln x}$$

which now looks familiar!

$$= \frac{1}{2}\int \left(\frac{1}{x}\right)\left(\frac{1}{\ln x}\right) dx$$

$$= \frac{1}{2}\ln\left|\ln x\right| + C$$

And now we can find some trig integrals that we were unable to do until now:

$$\int \tan x\,dx$$

Let $u = \cos x$

$du = -\sin x\,dx$

$$= \int \frac{\sin x}{\cos x}\,dx$$

$$= -\int \frac{du}{u}$$

$$= -\ln\left|u\right| + C$$

$$= -\ln\left|\cos x\right| + C$$

If you are "fancy" $= \ln\left|\cos x\right|^{-1} + C = \ln\left|\sec x\right| + C$

You try: $\int \cot x\,dx$

$$\int \tan(7x)\ dx$$

$$= \int \frac{\sin(7x)}{\cos(7x)} dx$$

$$\int \frac{\sec^2 x}{\tan x + 5}\ dx$$

A really weird integral!

$\int \sec x\, dx$ We can multiply by a clever form of one

CLEVER FORM OF ONE

$$= \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x}\, dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx$$

Let $u = \sec x + \tan x$

$du = [\sec x \tan x + \sec^2 x]\, dx$

$$= \int \frac{1}{u}\, du$$

$$= \ln|u| + C = \ln|\sec x + \tan x| + C$$

Definite Integrals [same as last chapter's]

$$\int_1^2 \frac{1-\cos\theta}{\theta - \sin\theta}\, d\theta$$

Let
$u = \theta - \sin\theta$
$du = (1-\cos\theta)\, d\theta$
$u(1) = 1 - \sin 1$
$u(2) = 2 - \sin 2$

$$= \int_{1-\sin 1}^{2-\sin 2} \frac{1}{u}\, du$$

$$= \ln|u| \Big|_{1-\sin 1}^{2-\sin 2}$$

$$= \ln|2-\sin 2| - \ln|1-\sin 1|$$

You try:

$$\int_1^e \frac{(1+\ln x)^2}{x}\,dx$$

$u = 1 + \ln x$

$du = \frac{1}{x}\,dx$

$u(1) = 1$

$u(e) = 2$

$$= \int_1^2 u^2\,du$$

$$= \left.\frac{u^3}{3}\right|_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

Let's not forget the Second FTC!

Let $F(x) = \int_1^x \frac{1}{t}\,dt$ Find $F'(x)$

$$F'(x) = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt$$

$$F'(x) = \frac{1}{x}$$

Let $F(x) = \int_1^{3x} \frac{1}{t}\,dt$ Find $F'(x)$

$$F'(x) = \frac{d}{dx}\int_1^{3x} \frac{1}{t}\,dt$$

$$= \left(\frac{1}{3x}\right)\left(\frac{d}{dx}(3x)\right) = \frac{1}{x}$$

Let $F(x) = \int_{\frac{\pi}{4}}^{x} \cot t\,dt$ Find $F'(x)$

$$F'(x) = \frac{d}{dx}\int_{\frac{\pi}{4}}^{x} \cot t\,dt = \cot x$$

Our first separable differential equation! SOLVE for y

Solve the following separable diff eq:

$$\frac{dy}{dx} = \frac{3}{2-x}$$ given: (1,0)is a point on the curve

$$dy = \frac{3}{2-x}\,dx$$ Separate! MULTIPLY BOTH SIDES by dx

$$\int dy = \int \frac{3}{2-x}\,dx$$ Integrate! BOTH SIDES

$$y + C_1 = -3\ln|2-x| + C_2$$ Combine the constants.

$$y = -3\ln|2-x| + C$$ Solve for C using the given point Point (1,0)

Let $x = 1,\ y = 0$

$$0 = -3\ln|2-1| + C$$ Hence $C = 0$

$$y = -3\ln|2-x|$$ Solve for y

You try:

$$\frac{dy}{dx} = \frac{2x}{x^2+1}$$

no given which means we don't HAVE TO SOLVE FOR C

$$dy = \frac{2x}{x^2+1}\,dx$$

$$\int dy = \int \frac{2x}{x^2+1}\,dx$$

$u = x^2+1$

$du = 2x\,dx$ $\int \frac{1}{u}\,du$

$$y = \ln(x^2+1) + C$$

Homework: pages 338 and 339 #2, 4, 6, 7, 10, 12, 20, 30, 34, 35, 36, 38, 40, 48, 50, 62, 64

Tuesday's homework [GROUNDHOG DAY]