Separable Differential Equations

$$\frac{dy}{dt} = \cos(2t) \qquad \text{Given: } y(0) = 1$$

$$dy = \cos(2t)\,dt$$

SEPARATE [MULTIPLY OR DIVIDE]

$$\int dy = \int \cos(2t)\,dt$$

integrate

$$y = \frac{1}{2}\sin(2t) + C$$

"+ C " must be here!

$$1 = \frac{1}{2}\sin(0) + C$$

hence, C=1

SOLVE FOR C

$$y = \frac{1}{2}\sin(2t) + 1$$

Our SOLUTION!!!

The solution to a separable differentiable equation is a differentiable function on an open interval which contains the initial given x- value.

The short answer is that the domain should be an open interval containing the initial condition, on which the solution is continuously differentiable. [See "diffeqspecial" at our website]

If no initial value is given, then your solution is the family of functions whose derivative is given as $\frac{dy}{dx}$. [Don't forget the "+C".]

Our four steps:
Separate
Integrate
Solve for C [or solve for y]
Solve for y [or solve for C]

Slightly harder: no initial value given

$$\frac{dy}{dx} = 4 - y$$

Be careful when separating!

$$\frac{dy}{4-y} = dx$$

SEPARATE

$$\int \frac{dy}{4-y} = \int dx$$

INTEGRATE

$4 - y > 0$

$$-\ln(4-y) = x + C$$

$$\ln(4-y) = -x + C$$

$$e^{\ln(4-y)} = e^{-x+C}$$

$$4 - y = Ce^{-x}$$

$$4 - Ce^{-x} = y$$

$$y' = \frac{5x}{y}$$

Given: (0, −1) is on the curve

$$\frac{dy}{dx} = \frac{5x}{y}$$

$$y\,dy = 5x\,dx$$

$$\int y\,dy = \int 5x\,dx$$

$$\frac{y^2}{2} = \frac{5x^2}{2} + C$$

use (0, −1) to solve for C

$$\frac{1}{2} = 0 + C \qquad C = \frac{1}{2}$$

$$\frac{y^2}{2} = \frac{5x^2}{2} + \frac{1}{2}$$

$$y^2 = 5x^2 + 1$$

Remember that y must contain (0, −1)

$$y = -\sqrt{5x^2+1}$$

needed

Really tricky!

$$\frac{dy}{dx} = xy + 2y$$

No given initial value

We must FACTOR!!!

$$\frac{dy}{dx} = y(x+2)$$

$$\frac{1}{y}\,dy = (x+2)\,dx$$

$$\int \frac{1}{y}\,dy = \int (x+2)\,dx$$

$$\ln y = \frac{x^2}{2} + 2x + C$$

$$e^{\ln y} = e^{\frac{x^2}{2}+2x+C}$$

$$y = Ce^{\frac{x^2}{2}+2x}$$

$y>0$

Remember e^C is A constant

You try:

$$\frac{dy}{dx} = \frac{x^3}{y^2}$$ Given: $y(2) = 3$

$$y^2\,dy = x^3\,dx$$

$$\int y^2\,dy = \int x^3\,dx$$

$$\frac{y^3}{3} = \frac{x^4}{4} + C$$

$$\frac{(3^3)}{3} = \frac{2^4}{4} + C$$ Hence $C = 5$

$$\frac{y^3}{3} = \frac{x^4}{4} + 5$$

$$y^3 = \frac{3x^4}{4} + 15$$

$$y = \sqrt[3]{\frac{3x^4}{4} + 15}$$

Here's a tricky one to try:

$$\frac{dy}{dx} = \frac{3x}{y}$$

Given: $y(6) = -4$

$$y\,dy = 3x\,dx$$

$$\int y\,dy = \int 3x\,dx$$

$$\frac{y^2}{2} = \frac{3x^2}{2} + C$$

$$\frac{(-4)^2}{2} = \frac{3(6^2)}{2} + C$$

$$8 = 54 + C \qquad C = -46$$

$$\frac{y^2}{2} = \frac{3x^2}{2} - 46$$

$$y^2 = 3x^2 - 92$$

$$y = -\sqrt{3x^2 - 92}$$

we need because y must contain (6, −4)

If time, try:

$$\frac{dy}{dx} = x^2\sqrt{y} \qquad \text{Given: } y(0) = 9$$

$$\frac{1}{\sqrt{y}}\,dy = x^2\,dx$$

$$\int \frac{1}{\sqrt{y}}\,dy = \int x^2\,dx$$

$$2\sqrt{y} = \frac{x^3}{3} + C$$

$$2\sqrt{9} = 0 + C \qquad C = 6$$

$$2\sqrt{y} = \frac{x^3}{3} + 6$$

$$\sqrt{y} = \frac{x^3}{6} + 3$$

$$y = \left(\frac{x^3}{6} + 3\right)^2$$

Common Applications:

Let $s(t)$ be the position function
We already know that

$$\frac{ds}{dt} = v(t)$$

$$ds = v(t)\ dt$$

$$\int ds = \int v(t)\ dt$$

$$s(t) = \int v(t)\,dt$$

Likewise:

$$v(t) = \int a\ (t)\,dt$$

Reminder:
Speed = $|\ v(t)\ |$

Total Distance Traveled on $[a,\ b] = \int_a^b |\ v(t)\ |\ dt$

Displacement on $[a,b] = \int_a^b v(t)\,dt$

REMEMBER IT IS SOMETIMES EASIER TO SOLVE FOR y FIRST

Homework: page 429 # 1, 5, 7, 23, 24