Growth and Decay Problems [an application of diff eq]

If y is a differentiable function of t such that $y > 0$ and $y' = ky$, then $y = Ce^{kt}$

C is the initial value [when $t = 0$]

k is the constant of proportionality

If $k > 0$, then exponential growth occurs.
If $k < 0$, then exponential decay occurs.

Why it works:

$y' = ky$

$\frac{dy}{dt} = ky$ Separate!

$\frac{dy}{y} = k\ dt$ Now integrate!

$\int \frac{dy}{y} = \int k\,dt$ $y > 0$ [given above]

$\ln y = kt + C$ $+C$ must appear here!

$e^{\ln y} = e^{(kt+C)}$ Solve for y

$y = e^{kt} \quad e^{C}$ That d a r n algebra trick!

$y = Ce^{kt}$ Since e^{C} is a constant

e^C is just a constant

Some problems to ponder:

The rate of change of y is proportional to y and when $t = 0,\ y = 2$ and when $t = 2,\ y = 4$. Find the value of y when $t = 3$. [This is exponential growth]

(t, y)

Translate into an equation:

$$\frac{dy}{dt} = ky$$

$$y = Ce^{kt}$$

Use our given information to find C and to find k

$(0, 2)$ $\quad 2 = Ce^0$ $\quad$ Hence, $C = 2$

(t, y)

$(2, 4)$ $\quad y = 2e^{kt}$

$$4 = 2e^{2k} \qquad \text{Solve for } k$$

$$2 = e^{2k} \qquad \text{Take the ln of both sides}$$

$$\ln 2 = \ln e^{2k}$$

$$\ln 2 = 2k$$

$$\frac{\ln 2}{2} = k$$

OUR CONSTANT OF PROPORTIONALITY

Now we can write our function

WHY NOT STORE "K" IN OUR TI!

$$y = 2e^{\frac{\ln 2}{2}t} \qquad \text{Now find } y(3)$$

$$y(3) = 2e^{\frac{\ln 2}{2}(3)} \approx 5.657$$

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.5ln(2)→K
          .3465735903
Y1(3)
          5.656854249
█
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Radioactive decay [think Rocky Flats!]

You found 10 grams of the plutonium isotope Pu-239 over by the former Rocky Flats and released it. How long will it take for the 10 grams to decay to 1 gram. The rate of decay is proportional to y. The half-life is 24000 years.

(t, g)

What two ordered pairs do we have? $(0, 10)$ $(24000, 5)$

$$\frac{dy}{dt} = ky$$

$$y = Ce^{kt}$$

$$y = 10e^{kt}$$

What is C? 10

Now use the other ordered pair to find k. Then find how long it takes to decay to 1 gram.

$$y = 10e^{-kt}$$

$$5 = 10e^{24000k}$$

$$\frac{1}{2} = e^{24000k}$$

$$\ln\frac{1}{2} = \ln e^{24000k}$$

$$\ln \frac{1}{2} = 24000k$$

$$\frac{\ln \frac{1}{2}}{24000} = k$$

$$y = 10e^{\frac{\ln \frac{1}{2}}{24000}t}$$

$$1 = 10e^{\frac{\ln \frac{1}{2}}{24000}t}$$

$$\ln .1 = \ln e^{\frac{\ln \frac{1}{2}}{24000}t}$$

$$\frac{\ln .1}{\frac{\ln \frac{1}{2}}{24000}} = t$$

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(ln(.5))/24000→K
        -2.88811325E-5
(ln(.1))/K
           79726.27428
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← a VERY LONG TIME

Compound [Continuous] Interest
[You probably did a lot of these before!]

Remember? $A = Pe^{rt}$ is A Form of $y = Ce^{kt}$

If you have \$1000 to invest at a rate of 6%, then how long will it take to double your investment?

$$A = Pe^{rt} \qquad P = 1000,\ r = .06,\ A = 2P \text{ or } 2000$$

$$2000 = 1000e^{.06t}$$

take ln of both side

$$2 = e^{.06t}$$

$$\ln 2 = \ln e^{.06t}$$

$$\ln 2 = .06t$$

$$\frac{\ln 2}{.06} = t$$

$$t \approx 11.552 \text{ years}$$

I could retire in 13 years. If I invested \$100000 in a certificate of deposit that has 6.5% interest, then how much money will I have when I retire?

$$A = Pe^{rt}$$

$$A = 100000e^{(.065)(13)}$$

$$A = \$232797.78$$

Mathionium has a half-life of 15 years. If you have 20 grams of Mathionium, then what amount will you have in 100 years? $(0,20),(15,10)$

$y = Ce^{kt}$ What is C? 20 Find k.

$y = 20e^{kt}$

$10 = 20e^{15k}$

$\frac{1}{2} = e^{15k}$

$\ln\frac{1}{2} = \ln e^{15k}$

$\ln\frac{1}{2} = 15k$

$\frac{\ln\frac{1}{2}}{15} = k$

$y = 20e^{\frac{\ln\frac{1}{2}}{15}t}$

$y(100) \approx .197$

The rate of change of P is proportional to P. When $t = 0$, $P = 5000$ and when $t = 1$, $P = 4750$. What is the value of P when $t = 5$?

(t, P)

$$\frac{dP}{dt} = kP$$

This is just a $y = Ce^{kt}$ problem.

$P = Ce^{kt}$

Newton's Law of Cooling [used by CSI!]
The rate of change in the temperature of an object in a room whose temperature is proportional to the difference between the object's temperature and the temperature of the surrounding medium.

From page 417 in our textbook:
Let y represent the temperature of an object in a room whose temperature is kept at a constant ***60 degrees***. If the object cools from 100 degrees to 90 degrees in 10 minutes, how much longer will it take for its temperature to decrease to 80 degrees? $80 \le y \le 100$

$$\frac{dy}{dt} = k(y-60)$$ Separate

$$\frac{dy}{y-60} = k\ dt$$ Integrate

$$\int \frac{dy}{y-60} = \int k\ dt$$

$$\ln(y-60) = kt + C$$

$$e^{\ln(y-60)} = e^{kt+C}$$

$$y-60 = Ce^{kt}$$ Use (0, 100)

$$100-60 = Ce^{0}$$ Hence, $C = 40$

$$y-60 = 40e^{kt}$$ Isolate y

$$y = 40e^{kt} + 60$$

$y = 40e^{kt} + 60$ Now use (10, 90) and solve for k

$90 = 40e^{10k} + 60$

$30 = 40e^{10k}$

$\frac{3}{4} = e^{10k}$

$\ln\left(\frac{3}{4}\right) = \ln e^{10k}$

$\ln\left(\frac{3}{4}\right) = 10k$

$-0.0287682072 \approx k$

Hence: $y \approx 40e^{-.02876t} + 60$

Now we can find when $y = 80$ degrees

$$80 = 40e^{-.02876t} + 60$$

$$\frac{1}{2} = e^{-.02876t}$$

$$\ln\frac{1}{2} = \ln e^{-.02876t}$$

$$\ln\left(\frac{1}{2}\right) = -.02876t$$

$$t \approx 24.1010418 \text{ minutes}$$

Homework: pages 418, 419 #21, 23, 33, 41, 44