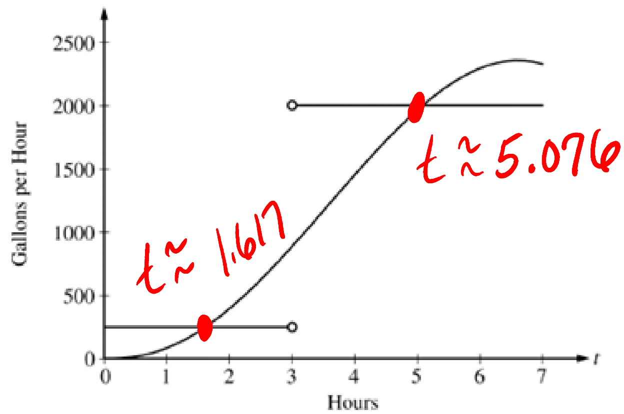

The amount of water in a storage tank, in gallons, is modeled by a continuous function on the time interval $0 \le t \le 7$, where t is measured in hours. In this model, rates are given as follows:

(i) The rate at which water enters the tank is $f(t) = 100t^2 \sin(\sqrt{t})$ gallons per hour for $0 \le t \le 7$.

(ii) The rate at which water leaves the tank is

$$g(t) = \begin{cases} 250 & \text{for } 0 \le t < 3 \\ 2000 & \text{for } 3 < t \le 7 \end{cases} \text{ gallons per hour.}$$

The graphs of f and g, which intersect at $t = 1.617$ and $t = 5.076$, are shown in the figure above. At time $t = 0$, the amount of water in the tank is 5000 gallons.

(a) How many gallons of water enter the tank during the time interval $0 \le t \le 7$? Round your answer to the nearest gallon.

$$\int_0^7 f(t)\ dt \approx 8264 \text{ gallons}$$

(b) For $0 \le t \le 7$, find the time intervals during which the amount of water in the tank is decreasing. Give a reason for each answer.

The amount of water [wudder] in the tank is decreasing on the intervals $0 \le t \le 1.617$ and $3 \le t \le 5.076$ because during those time intervals $f(t) < g(t)$. [I looked at the graph and found where the rate at which water entering the tank is less than the rate at which water is leaving the tank]

(c) For $0 \le t \le 7$, at what time t is the amount of water in the tank greatest? To the nearest gallon, compute the amount of water at this time. Justify your answer.

[Ack! This is a closed interval so we must remember to check the endpoints!]

[First we need to find any candidates other than the endpoints.]
At time $t = 3$ $f(t) - g(t)$ changes from positive to negative values.
[Now to find the amount of water in the tank at time $t = 0$, $t = 3$, $t = 7$]

Amount at time $t = 0$ is 5000 gallons [given]

Amount at time $t=3$

$$5000+\int_0^3 f(t)\ dt - \int_0^3 250\ dt \approx 5126.591 \text{ gallons}$$

[Initial amount + amount that entered – amount that left the tank

Amount of water in the tank at $t=7$

$$5126.591+\int_3^7 f(t)dt-\int_3^7 2000\,dt \approx 4513.807 \text{ gallons}$$

[Amount at time $t = 3$ + amount that entered during $3 \le t \le 7$ minus the amount that left the tank during $3 \le t \le 7$]

Hence, the amount of water is the greatest at time $t=3$ hours. At that time, the amount of water in the tank to the nearest gallon is 5127 gallons.

[Note: Yes, they did penalize if you did not have your answer to the "nearest gallon"]

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x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	−4	4	2
4	−1	3	6	7

The functions f and g are differentiable for all real numbers, and g is strictly increasing. The table above gives values of the functions and their first derivatives at selected values of x. The function h is given by $h(x) = f(g(x)) - 6$.

(a) Explain why there must be a value r for $1 < r < 3$ such that $h(r) = -5$.

$$h(1) = f(g(1)) - 6 = 3$$
$$h(3) = f(g(3)) - 6 = -7$$

Since h is continuous, then by the Intermediate Value Theorem there exists a value $r, 1 < r < 3$ such that $-7 < h(r) < 3$ and since $-7<-5<3$ then there must be a value $r, 1 < r < 3$ such that $h(r) = -5$

(b) Explain why there must be a value c for $1 < c < 3$ such that $h'(c) = -5$.

By the Mean Value Theorem there is a value c, $1 < c < 3$, such that $h'(c) = \frac{h(3) - h(1)}{3-1}$ which equals $\frac{-7-3}{3-1} = -5$. Hence, there is a value c, $1 < c < 3$ such that $h'(c) = -5$.

(c) Let w be the function given by $w(x) = \int_1^{g(x)} f(t)\,dt$. Find the value of $w'(3)$.

$$w'(x) = \frac{d}{dx}\int_1^{g(x)} f(t)\,dt = f(g(x))\frac{d}{dx}g(x)$$

So, $w'(x) = f(g(x))g'(x)$

Then, $w'(3) = f(g(3))g'(3) = -2$

(d) If g^{-1} is the inverse function of g, write an equation for the line tangent to the graph of $y = g^{-1}(x)$ at $x = 2$.

[What do we need? A point and a slope]

To find the point: $g(1) = 2$ so $g^{-1}(2) = 1$

To find the slope: $\left(g^{-1}\right)'(2) = \frac{1}{g'(1)} = \frac{1}{5}$

So, our point is $(2,1)$ and the slope of the tangent line is $\frac{1}{5}$

Hence, the equation of the tangent line is $y - 1 = \frac{1}{5}(x-2)$

The rate at which people enter an auditorium for a rock concert is modeled by the function R given by $R(t) = 1380t^2 - 675t^3$ for $0 \le t \le 2$ hours; $R(t)$ is measured in people per hour. No one is in the auditorium at time $t = 0$, when the doors open. The doors close and the concert begins at time $t = 2$.

(a) How many people are in the auditorium when the concert begins?

$\int_0^2 R(t)\,dt \approx 980$ people [you can't have a partial person!]

(b) Find the time when the rate at which people enter the auditorium is a maximum. Justify your answer.

[First find the critical values]

$R'(t) = 0$ when $t = 0$ and $t \approx 1.36296$

Let $A = 1.36296$

[Remember that you must check the endpoints!]

$R(0) = 0$

$R(A) = 854.527$

$R(2) = 120$

Hence, the maximum rate occurs when $t \approx 1.36296$

(c) The total wait time for all the people in the auditorium is found by adding the time each person waits, starting at the time the person enters the auditorium and ending when the concert begins. The function w models the total wait time for all the people who enter the auditorium before time t. The derivative of w is given by $w'(t) = (2 - t)R(t)$. Find $w(2) - w(1)$, the total wait time for those who enter the auditorium after time $t = 1$.

$$w(2) - w(1) = \int_1^2 w'(t)\, dt$$

$$= \int_1^2 (2-t)(R(t))dt$$

$$\approx 387.5$$

Hence, the total wait time for those who enter the auditorium after time $t = 1$ hour is 387.5 hours.

(d) On average, how long does a person wait in the auditorium for the concert to begin? Consider all people who enter the auditorium after the doors open, and use the model for total wait time from part (c).

[This one is really tricky!]

$$\frac{1}{980}(w(2)) = \frac{1}{980}\int_0^2 (2-t)(R(t))dt$$

$$= 0.77551$$

Hence, on average, a person waits 0.77551 hours.

The non-calculator free response

x	2	3	5	8	13
$f(x)$	1	4	–2	3	6

Let f be a function that is twice differentiable for all real numbers. The table above gives values of f for selected points in the closed interval $2 \le x \le 13$.

(a) Estimate $f'(4)$. Show the work that leads to your answer.

$$f'(4) \approx \frac{f(5) - f(3)}{5-3} = -3$$

[Yes, you must have the difference quotient]

(b) Evaluate $\int_2^{13} (3 - 5f'(x))\, dx$. Show the work that leads to your answer.

[Remember to use the rules of integration!]

$$\int_2^{13} (3 - 5f'(x))\, dx = \int_2^{13} 3\, dx \;-5 \int_2^{13} f'(x)\, dx$$

$$= 3x \Big|_2^{13} - 5\left[f(x) \Big|_2^{13} \right]$$

$$= [39 - 6] - 5[f(13) - f(2)]$$

$$= 8$$

(c) Use a left Riemann sum with subintervals indicated by the data in the table to approximate $\int_2^{13} f(x)\, dx$. Show the work that leads to your answer.

$$\int_2^{13} f(x)\, dx \approx (1)(f(2)) + (2)(f(3)) + (3)(f(5)) + (5)(f(8)) = 18$$

[Remember to show that you are using function values for the heights of your rectangles]

(d) Suppose $f'(5) = 3$ and $f''(x) < 0$ for all x in the closed interval $5 \le x \le 8$. Use the line tangent to the graph of f at $x = 5$ to show that $f(7) \le 4$. Use the secant line for the graph of f on $5 \le x \le 8$ to show that $f(7) \ge \frac{4}{3}$.

The equation of the tangent line at $x = 5$ is: $y + 2 = 3(x - 5)$

Since $f''(x) < 0$, then the graph of f is concave down and the tangent line will lie above the curve for all $x,\ 5 \le x \le 8$.

To estimate $f(7): y+2=3(7-5)$, hence $f(7)=4$ which would mean that $f(7)\le 4$ [because the tangent line is above the curve]

First find the slope of the secant line: $m_{\sec}=\dfrac{f(8)-f(5)}{8-5}=\dfrac{5}{3}$

The equation of the secant line between $x=5$ and $x=8$: $y-3=\dfrac{5}{3}(x-8)$

OR $y+2=\dfrac{5}{3}(x-5)$

Since $f''(x)<0$ for x, x, $5\le x\le 8$, the secant line lies below the graph of $f(x)$.

To estimate $f(7)$ using the secant line: $y+2=\dfrac{5}{3}(7-5)$

$y=\dfrac{4}{3}$

Since the secant line is below the graph, then $f(7)\ge\dfrac{4}{3}$

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t (seconds)	0	8	20	25	32	40
$v(t)$ (meters per second)	3	5	−10	−8	−4	7

The velocity of a particle moving along the x-axis is modeled by a differentiable function v, where the position x is measured in meters, and time t is measured in seconds. Selected values of $v(t)$ are given in the table above. The particle is at position $x=7$ meters when $t=0$ seconds.

(a) Estimate the acceleration of the particle at $t=36$ seconds. Show the computations that lead to your answer. Indicate units of measure.

$a(t)=v'(t)$

$$a(36)\approx\frac{v(40)-v(32)}{40-32}$$

$$=\frac{7-(-4)}{8}$$

$$=\frac{11\ meters}{8(\sec)^2}$$

(b) Using correct units, explain the meaning of $\int_{20}^{40} v(t)\,dt$ in the context of this problem. Use a trapezoidal sum with the three subintervals indicated by the data in the table to approximate $\int_{20}^{40} v(t)\,dt$.

$\int_{20}^{40} v(t)\, dt$ is the particle's change in position in meters from time $t = 20$ seconds to time $t = 40$ seconds.

$$\int_{20}^{40} v(t)\, dt \approx \frac{v(20)+v(25)}{2}\cdot 5 + \frac{v(25)+v(32)}{2}\cdot 7 + \frac{v(32)+v(40)}{2}\cdot 8$$
$$= -75 \text{ meters}$$

(c) For $0 \le t \le 40$, must the particle change direction in any of the subintervals indicated by the data in the table? If so, identify the subintervals and explain your reasoning. If not, explain why not.

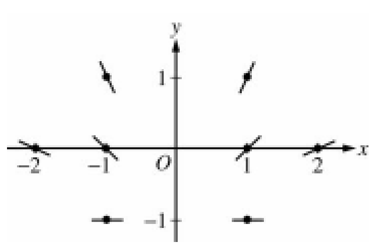
$v(8) > 0$ and $v(20) < 0$
$v(32) < 0$ and $v(40) > 0$
Therefore, the particle changes direction in the intervals
$8 < t < 20$ and $32 < t < 40$.

Suppose that the acceleration of the particle is positive for $0 < t < 8$ seconds. Explain why the position of the particle at $t = 8$ seconds must be greater than $x = 30$ meters.

Since $v'(t) = a(t) > 0$ for $0 < t < 8$, $v(t) \ge 3$ on this interval.
Therefore, $x(8) = x(0) + \int_0^8 v(t)\, dt \ge 7 + 8 \cdot 3 > 30$.

Consider the differential equation $\frac{dy}{dx} = \frac{1+y}{x}$, where $x \neq 0$.

(a) On the axes provided, sketch a slope field for the given differential equation at the eight points indicated.



(b) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(-1) = 1$ and state its domain.

$$\frac{dy}{dx} = \frac{1+y}{x}$$

$$\frac{1}{1+y}\,dy = \frac{1}{x}\,dx$$

$$\int \frac{1}{1+y}\,dy = \int \frac{1}{x}\,dx$$

$$\ln|1+y| = \ln|x| + C_1$$

[you must have ab val]

$$e^{\ln|1+y|} = e^{\ln|x|+C_1}$$

$$|1+y| = C|x|$$

NOTE:

$$|1+y| = \begin{cases} -(1+y), & y < -1 \\ 1+y, & y \geq -1 \end{cases}$$

$$|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

BUT $x \neq 0$

(-1,1) Given

$$1+y = C(-x)$$

$$1+1 = C(-(-1))$$

$$2 = C$$

$$1+y = -2x$$

$$y = -2x-1$$

Domain: $y = -2x-1$ for $x < 0$

[because our given point was (-1,1)]