Free Response #1

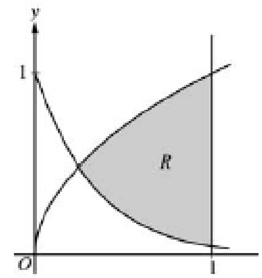
Question 1

Let R be the shaded region bounded by the graphs of $y = \sqrt{x}$ and $y = e^{-3x}$ and the vertical line $x = 1$, as shown in the figure above.

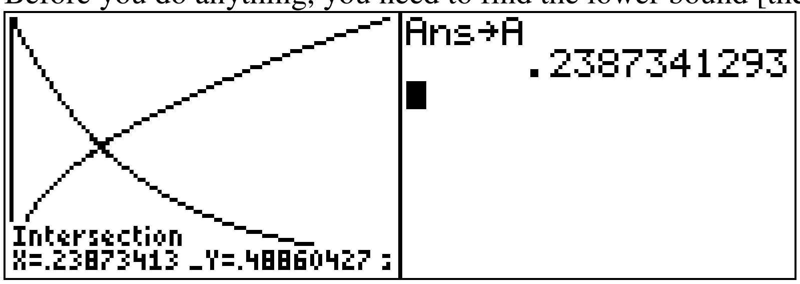
(a) Find the area of R.

(b) Find the volume of the solid generated when R is revolved about the horizontal line $y = 1$.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a rectangle whose height is 5 times the length of its base in region R. Find the volume of this solid.



Before you do anything, you need to find the lower bound [the upper bound is x = 1]



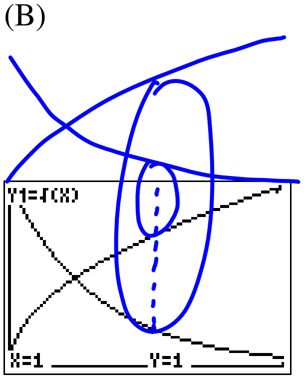
$\sqrt{x} = e^{-3x}$ at $x = A$ where A = .2387341293 [our lower bound]

(A) Area of R = $\int_A^1 \left[\sqrt{x} - e^{-3x}\right] dx$

$\approx 0.442 \ or \ 0.443$

(B)

We need R(x) and r(x)



$R(x) = 1 - e^{-3x}$ and $r(x) = 1 - \sqrt{x}$

Hence, Volume = $\pi \int_A^1 \left[\left(1 - e^{-3x}\right)^2 - \left(1 - \sqrt{x}\right)^2 \right] dx$

$= 0.453\pi$

(C) Area of a cross section = $A(x) = b\,h$ but the height is five times the base so

$A(x) = 5b^2$ where the base is equal to the distance between the two curves

So, $A(x) = 5\left(\sqrt{x} - e^{-3x}\right)$

Hence, Volume = $\int_A^1 5\left(\sqrt{x} - e^{-3x}\right)^2\, dx$

≈ 1.554

A particle moves along the x-axis so that its velocity at time t is given by

$$v(t) = -(t+1)\sin\left(\frac{t^2}{2}\right).$$

At time $t = 0$, the particle is at position $x = 1$.

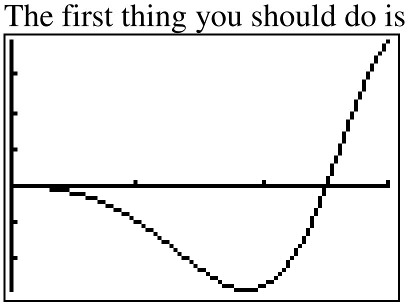
(a) Find the acceleration of the particle at time $t = 2$. Is the speed of the particle increasing at $t = 2$? Why or why not?

(b) Find all times t in the open interval $0 < t < 3$ when the particle changes direction. Justify your answer.

(c) Find the total distance traveled by the particle from time $t = 0$ until time $t = 3$.

(d) During the time interval $0 \le t \le 3$, what is the greatest distance between the particle and the origin? Show the work that leads to your answer.

The first thing you should do is to graph the velocity graph to get an idea of what is going on.



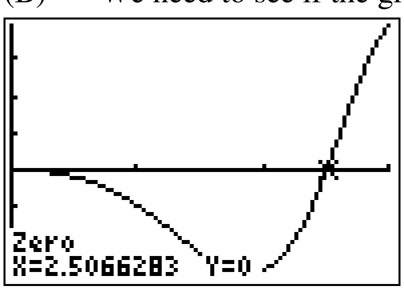
$v(t)$ *on* $[0, 3]$

(A) $a(t) = v'(t)$ so $a(2) = v'(2) \approx 1.587$ *or* 1.588

$v(2) \approx -2.7278$

Since $v(2) < 0$ and $a(2) > 0$, then the speed is deceasing at $t = 2$

(B) We need to see if the graph of $v(t)$ crosses the $t-$axis on the interval



At $t \approx 2.5066283$ the graph of $v(t)$ changes from negative to positive values. Hence the particle changes direction at $t \approx 2.5066283$

(C) Total distance traveled on [0, 3] = $\int_0^3 |v(t)|\, dt \approx 4.333$ or 4.334

Here is what I did to find it: [but I would NOT write the calculator stuff down]

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fnInt(abs(Y1),X,
0,3)
        4.333818626
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(D) Remember that absolute extrema may occur at either the endpoints or at a critical value. We need to find the displace at these values.

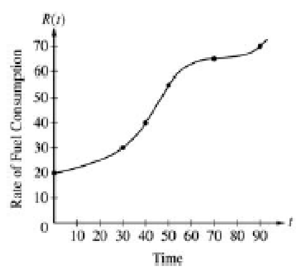
$x(0) = 1$ [initial position was given]

$$x(3) = x(0) + \int_0^3 v(t)\, dt \approx -2.197 + 1$$

$$x(2.5066283) = x(0) + \int_0^{2.5066283} v(t)\, dt \approx -3.2654 + 1$$

Hence, the greatest distance from the origin is 2.2654 [on the left of the origin]

The rate of fuel consumption, in gallons per minute, recorded during an airplane flight is given by a twice-differentiable and strictly increasing function R of time t. The graph of R and a table of selected values of $R(t)$, for the time interval $0 \le t \le 90$ minutes, are shown above.



t (minutes)	$R(t)$ (gallons per minute)
0	20
30	30
40	40
50	55
70	65
90	70

(a) Use data from the table to find an approximation for $R'(45)$. Show the computations that lead to your answer. Indicate units of measure.

(b) The rate of fuel consumption is increasing fastest at time $t = 45$ minutes. What is the value of $R''(45)$? Explain your reasoning.

(c) Approximate the value of $\int_0^{90} R(t)\, dt$ using a left Riemann sum with the five subintervals indicated by the data in the table. Is this numerical approximation less than the value of $\int_0^{90} R(t)\, dt$? Explain your reasoning.

(d) For $0 < b \le 90$ minutes, explain the meaning of $\int_0^b R(t)\, dt$ in terms of fuel consumption for the plane. Explain the meaning of $\frac{1}{b}\int_0^b R(t)\, dt$ in terms of fuel consumption for the plane. Indicate units of measure in both answers.

(A) $$R'(45) = \frac{R(50) - R(40)}{50 - 40} = 1.5 \frac{gal}{(\min)^2}$$

(B) Since $R'(45)$ is a maximum, then $R''(45) = 0$

(C)

$$\int_0^{90} R(t)\,dt \approx (30)(20) + (10)(30) + (10)(40)$$
$$+(20)(55) + (20)(65) = 3700$$

Yes, this approximation is less because the graph of R is increasing on the interval.

(D) $\int_0^b R(t)\,dt$ is the total amount of gallons of fuel used during the time interval $0 \le t \le b$

$\frac{1}{b}\int_0^b R(t)\,dt$ is the average value of the rate of fuel consumption in gallons/minute during the time interval $0 \le t \le b$

Non-calculator free response solutions

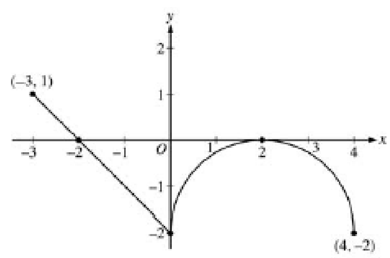
Let f be a function defined on the closed interval $-3 \le x \le 4$ with $f(0) = 3$. The graph of f', the derivative of f, consists of one line segment and a semicircle, as shown above.

(a) On what intervals, if any, is f increasing? Justify your answer.

(b) Find the x-coordinate of each point of inflection of the graph of f on the open interval $-3 < x < 4$. Justify your answer.

(c) Find an equation for the line tangent to the graph of f at the point $(0, 3)$.

(d) Find $f(-3)$ and $f(4)$. Show the work that leads to your answers.



Graph of f'

(A)

The function f is increasing on $[-3, -2]$ since $f' > 0$ for $-3 \le x < -2$.

(B) At $x = 0$ the graph of f' changes from decreasing to increasing so f has a point of inflection at $x = 0$. At $x = 2$ the graph of f' changes from increasing to decreasing so f has a point of inflection at $x = 2$.

(C) Point: (0, 3)
Slope of tangent line = $f'(0) = -2$
Equation of tangent line at $x = 0$ is: $y - 3 = -2(x - 0)$

(D) $f(-3) = f(0) + \int_{0}^{-3} f'(t)\, dt$

$= f(0) - \int_{-3}^{0} f'(t)\, dt$

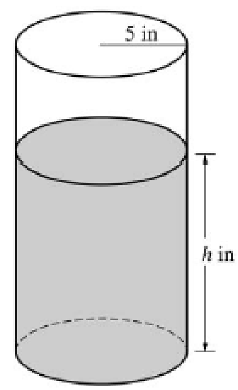
$= 3 - (0.5 - 2)$

$= 4.5$

$f(4) = f(0) + \int_{0}^{4} f'(t)\, dt$

$= 3 + \quad -\left(8 - \frac{1}{2}\left(2^2\right)\pi\right)$

A coffeepot has the shape of a cylinder with radius 5 inches, as shown in the figure above. Let h be the depth of the coffee in the pot, measured in inches, where h is a function of time t, measured in seconds. The volume V of coffee in the pot is changing at the rate of $-5\pi\sqrt{h}$ cubic inches per second. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)



(a) Show that $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$.

(b) Given that $h = 17$ at time $t = 0$, solve the differential equation $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$ for h as a function of t.

(c) At what time t is the coffeepot empty?

Things to notice: $r = 5$ inches so $\frac{dr}{dt} = 0$

(a) $V = \pi r^2 h$

$\frac{d}{dt} V = \frac{d}{dt}\left(\pi r^2 h\right)$

$\frac{dV}{dt} = 2\pi r \frac{dr}{dt}(h) + \pi r^2 \frac{dh}{dt}$

$-5\pi\sqrt{h} = 0 + \pi r^2 \frac{dh}{dt}$

$-5\sqrt{h} = 5^2 \frac{dh}{dt}$

$\frac{-\sqrt{h}}{5} = \frac{dh}{dt}$

(B) $\frac{-\sqrt{h}}{5} = \frac{dh}{dt}$

$-\frac{1}{5}\,dt = \frac{1}{\sqrt{h}}\,dh$

$\int -\frac{1}{5}\,dt = \int h^{-\frac{1}{2}}\,dh$

$-\frac{1}{5}t + C = 2\sqrt{h}$ $(0, 17)$

$0 + C = 2\sqrt{17}$ so $C = 2\sqrt{17}$

$-\frac{1}{5}t + 2\sqrt{17} = 2\sqrt{h}$

$-\frac{1}{10}t + \sqrt{17} = \sqrt{h}$

$\left(-\frac{1}{10}t + \sqrt{17}\right)^2 = h$

(c) The coffeepot will be empty when $h = 0$

$0 = -0.1t + \sqrt{17}$

$\sqrt{17} = 0.1t$

$10\sqrt{17} = t$

Let f be the function defined by

$$f(x) = \begin{cases} \sqrt{x+1} & \text{for } 0 \le x \le 3 \\ 5 - x & \text{for } 3 < x \le 5. \end{cases}$$

(a) Is f continuous at $x = 3$? Explain why or why not.

(b) Find the average value of $f(x)$ on the closed interval $0 \le x \le 5$.

(c) Suppose the function g is defined by

$$g(x) = \begin{cases} k\sqrt{x+1} & \text{for } 0 \le x \le 3 \\ mx + 2 & \text{for } 3 < x \le 5, \end{cases}$$

where k and m are constants. If g is differentiable at $x = 3$, what are the values of k and m?

(a) $f(3) = \sqrt{3+1}$

$\lim_{x \to 3^-} f(x) = 2$

$\lim_{x \to 3^+} f(x) = 2$

$\lim_{x \to 3} f(x) = 2 = f(3)$

Hence, f is continuous at $x = 3$

(b) Average value on [0, 5] $= \frac{1}{5-0}\int_0^5 f(x)\,dx$

$$\int_0^5 f(x)dx = \int_0^3 f(x)\,dx + \int_3^5 f(x)\,dx$$

$$\int_0^3 (x+1)^{\frac{1}{2}}\,dx + \int_3^5 (5-x)\,dx$$

$$= \frac{2}{3}(x+1)^{\frac{3}{2}}\Big|_0^3 + \left(5x - \frac{x^2}{2}\right)\Big|_3^5$$

$$= \frac{26}{3}$$

So, average value is $\frac{1}{5}\int_0^5 f(x)\,dx = \left(\frac{1}{5}\right)\left(\frac{20}{3}\right) = \frac{4}{3}$

(c)

Suppose the function g is defined by

$$g(x) = \begin{cases} k\sqrt{x+1} & \text{for } 0 \le x \le 3 \\ mx + 2 & \text{for } 3 < x \le 5, \end{cases}$$

where k and m are constants. If g is differentiable at $x = 3$, what are the values of k and m?

To be continuous:

$$g(3) = \lim_{x\to 3} g(x)$$

$$g(3) = 2k$$

$$\lim_{x\to 3^-} g(x) = 2k \quad \text{and} \quad \lim_{x\to 3^+} g(x) = 3m + 2$$

We need $2k = 3m + 2$

To be differentiable:

$$g'(x) = \begin{cases} \frac{k}{2\sqrt{x+1}} & 0 \le x \le 3 \\ m & 3 < x \le 5 \end{cases}$$

$$\lim_{x\to 3^-} g'(x) = \frac{k}{4} \quad \text{and} \quad \lim_{x\to 3^+} g'(x) = m$$

We need $m = \frac{k}{4}$ or $4m = k$

So, $2(4m) = 3m + 2$

$8m = 3m + 2$

$5m = 2$

$m = \frac{2}{5}$, $k = \frac{8}{5}$