Mean Value Theorem [another existence theorem]

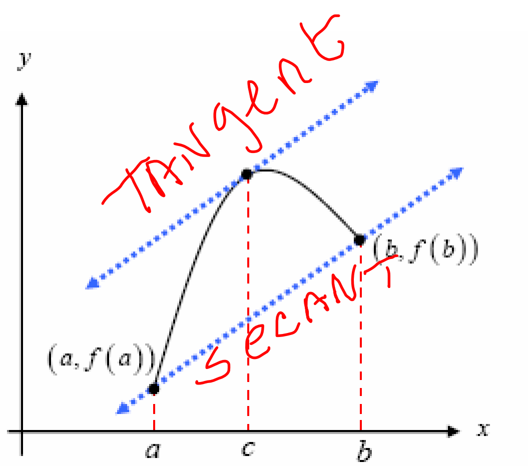
If f is continuous on the closed interval $[a,b]$ AND differentiable on the open interval (a,b), then there exists a number c in (a,b) such that

DIFFERENCE QUOTIENT

$$f'(c) = \frac{f(b) - f(a)}{b-a} \qquad \text{What?!}$$

Translation One:
If we draw a secant between the endpoints of a closed interval, then there exists at least one point in the interior of the interval where the slope of the secant will equal the slope of a tangent line.
Translation Two:
The instantaneous rate of change must equal the average rate of change at least once in the interior of the interval.

Let's look at a graphical representation:
From: http://chaoticgolf.com

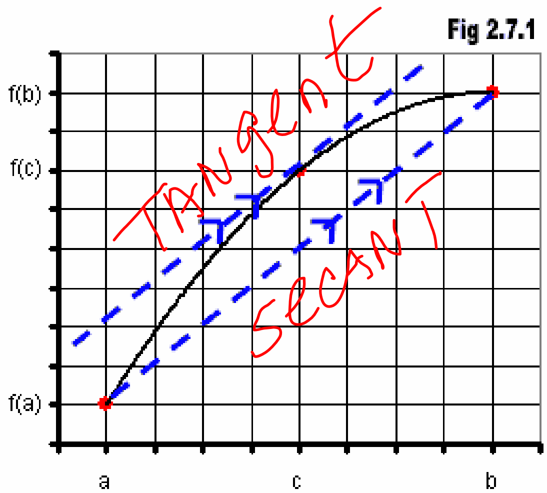


$m_{sec} = m_{TAN}$

AVERAGE RATE OF Δ

INST. RATE of Δ

Here's a graphical representation of this concept:



You are probably wondering why this is useful. Let's look at an example. Let's say a highway patrol officer notices a seemingly bad driver begin to break as he sees the police cruiser on the side of the road. The officer reads 50 mph on his radar where the speed limit is 60 mph. He calls another police officer 15 miles down the road and asks her to measure the time since his call to when that car approaches her cruiser. Exactly 12 minutes later, the car passes by. The reading on her radar also shows 50 mph. The officer pulled over the driver and gave him a speeding ticket.

Why did she give the driver a ticket if both officers saw that the driver was driving within the bounds of the speed limit? The answer to this question comes from the Mean Value Theorem. Let's solve for the average speed of the car:

$$\frac{15\text{ miles}}{12\cancel{\text{min}}}\times\frac{60\cancel{\text{min}}}{\text{hr}}=75\frac{\text{miles}}{\text{hr}}=75\text{ mph}$$

If the average speed is 75 mph, then at some point between the path of those 2 cruisers, The speed of the car reached 75 mph, meaning the car went over the speed limit of 60 mph.

[I can't remember where this example came from. Maybe Mr. Zab's webpage?]

This can happen more than once or just once. Caution: the function must meet the criteria above.

Let's try some MVT problems:

Page 177 #40 and 42 [Find the value that satisfies the MVT]

40. $f(x) = x\left(x^2 - x - 2\right)$ on [-1, 1]

$f(x) = x^3 - x^2 - 2x$

By MVT there is a c, $-1 < c < 1$, such that $f'(c) = \dfrac{f(1) - f(-1)}{1 - (-1)}$

$$f'(c) = \frac{0 - 2}{1 - (-1)} = -1$$

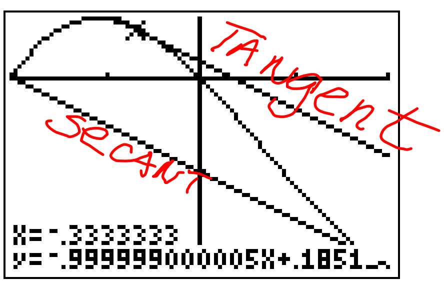
$$f'(x) = 3x^2 - 2x - 2$$

$$-1 = 3c^2 - 2c - 2$$

$$0 = 3c^2 - 2c - 1$$

$$0 = (3c+1)(c-1)$$

$$c = -\tfrac{1}{3} \qquad \cancel{c = 1}$$



$m_{sec}\Big|_{\text{on } [-1,1]} = m_{TAN}\Big|_{x = -\frac{1}{3}}$

42. $f(x) = \dfrac{x+1}{x}$ on $\left[\dfrac{1}{2}, 2\right]$

$f(x) = 1 + x^{-1}$

By MVT there is a c, $\frac{1}{2} < c < 2$, such that $f'(c) = \dfrac{f(2) - f(\frac{1}{2})}{2 - \frac{1}{2}}$

$f'(c) = \dfrac{1.5 - 3}{2 - \frac{1}{2}} = -1$

$-1 = -\dfrac{1}{c^2}$

1 or $\cancel{-1} = c$

Now let's consider some real-life problems:
Page 177 #58
When an object is removed from a furnace and placed in an environment with a constant temperature of 90°F, its core temperature is 1500°F. Five hours later the core temperature is 390°F. Explain why ~~there must exist~~ a time in the interval when the temperature is decreasing at a rate of 222°F per hour.

let $f(t)$ be the temp of
of object at time t
$f(5) = 390°F$ $\quad f(0) = 1500°F$
By the MVT there is a t, $0 < t < 5$,
such that $f'(t) = \dfrac{f(5) - f(0)}{5 - 0}$

so, $f'(t) = \dfrac{390 - 1500}{5 - 0} \dfrac{°F}{hr}$

$= -222 \dfrac{°F}{hr}$

Hence, there is a t $\; 0 < t < 5$
such that the temp is decreasing
at a rate of $222 \dfrac{°F}{hr}$

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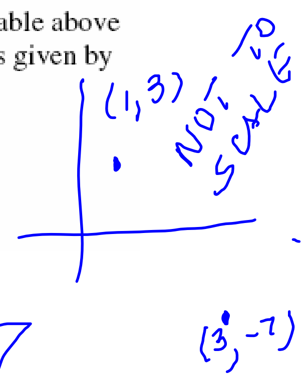
x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	-4	4	2
4	-1	3	6	7

$h(x) = f(g(x)) - 6$

3. The functions f and g are differentiable for all real numbers, and g is strictly increasing. The table above gives values of the functions and their first derivatives at selected values of x. The function h is given by $h(x) = f(g(x)) - 6$.

(a) Explain why there must be a value r for $1 < r < 3$ such that $h(r) = -5$.

(b) Explain why there must be a value c for $1 < c < 3$ such that $h'(c) = -5$.



$h(1) = f(g(1)) - 6 = 3$

$h(3) = f(g(3)) - 6 = -7$

By IVT there is an r, $1 < r < 3$, SUCH THAT $h(1) > h(r) > h(3)$

So there is an r, $1 < r < 3$ SUCH THAT $-7 < h(r) < 3$

Hence there must be an $h(r) = -5$ on $1 < r < 3$

By MVT there is a c $1 < c < 3$, SUCH THAT $h'(c) = \dfrac{h(3) - h(1)}{3 - 1}$

$$h'(c) = \frac{-7 - 3}{3 - 1} = -5$$

Hence, $h'(c) = -5$ for c, $1 < c < 3$

Homework: page 177 #39, 41, 43, 45, 57