Riemann Sums
(sung to the tune of Jingle Bells)

Riemann Sums, Riemann Sums
Counting Areas
Of rectangles whose widths get small
We need to count them all
Riemann Sums, Riemann Sums
Counting Areas
Of rectangles whose widths get small
We need to count them all.

We learn to integrate
It's really lots of fun.
It's easier to find
Than those old Riemann Sums
We learn to sub a u
When things get sort of hard
But most of all we tabulate
When we get sick of parts.
[repeat the refrain]

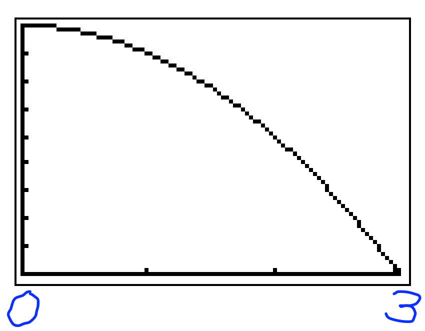
Summary of what we know so far:

We can estimate a definite integral by using a Riemann Sum.
LRAM is the left-hand rectangular approximation
RRAM is the right-hand rectangular approximation
MRAM is the mid-point rectangular approximation

If a function is ***increasing and concave up***, then the LRAM is an under-approximation of the actual area; the RRAM is an over-approximation of the actual area; and the MRAM is an under-approximation of the actual area.

If a function is ***increasing and concave down***, then the LRAM is an under-approximation of the actual area; the RRAM is an over-approximation of the actual area; and the MRAM is an over-approximation of the actual area.
Now for some new stuff:

Let's consider a decreasing function.
Let $y = 9 - x^2$ on $[0, 3]$



6 rectangles
$dx = \text{width} = \frac{1}{2}$

Find an approximation of $\int_0^3 \left(9-x^2\right) dx$ using LRAM, RRAM, and MRAM with ***six*** subintervals

$$\int_0^3 \left(9-x^2\right) dx \approx LRAM$$

$LRAM = \frac{1}{2}\left[f(0)+f\left(\frac{1}{2}\right)+f(1)+f(1.5)+f(2)+f(2.5)\right]$

$LRAM = \frac{1}{2}\left[9+8.75+8+6.75+5+2.75\right]$

$= 20.125$

$$\int_0^3 \left(9-x^2\right) dx \approx RRAM$$

$RRAM = \frac{1}{2}\left[f\left(\frac{1}{2}\right)+f(1)+f(1.5)+f(2)+f(2.5)+f(3)\right]$

$= \frac{1}{2}\left[8.75+8+6.75+5+2.75+0\right]$

$= 15.625$

$$\int_0^3 \left(9-x^2\right) dx \approx MRAM$$

$MRAM = \frac{1}{2}\left[f(.25)+f(.75)+f(1.25)+f(1.75)+f(2.25)+f(2.75)\right]$

$= 18.836$

Since $\int_0^3 \left(9 - x^2\right) dx = 18$, then

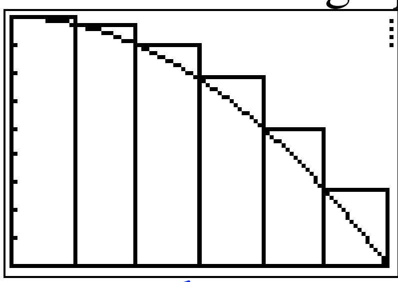
LRAM was an __OVER__ - approximation

RRAM was an __UNDER__ - approximation

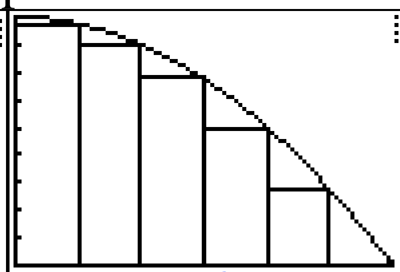
MRAM was an __OVER__ - approximation

We should probably remember this for any function that is ***decreasing and concave down***.

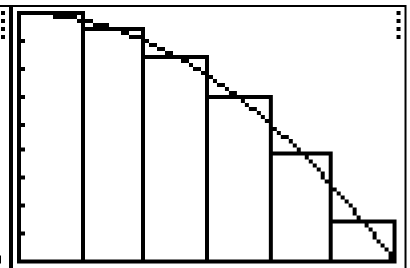
Here are some graphs for our Riemann Sums:



LRAM



RRAM

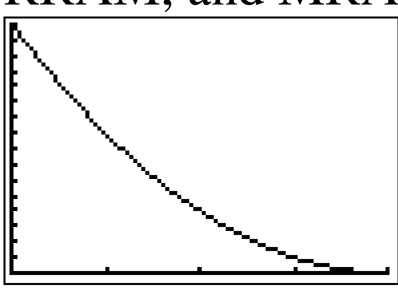


MRAM

Now let's consider a ***decreasing and concave up*** function.

Let $f(x) = (x-4)^2$ on [0, 4]

Find an approximation for $\int_0^4 (x-4)^2\, dx$ using LRAM, RRAM, and MRAM using 8 sub-intervals



$\frac{d}{dx}$ = WIDth = $\frac{1}{2}$

$$\int_0^4 (x-4)^2\,dx \approx \text{LRAM}$$

$$\text{LRAM} = \frac{1}{2}\left[f(0)+f\left(\tfrac{1}{2}\right)+f(1)+f(1.5)+f(2)+f(2.5)+f(3)+f(3.5)\right]$$

$$= 25.375$$

$$\int_0^4 (x-4)^2\,dx \approx \text{RRAM}$$

$$\text{RRAM} = \frac{1}{2}\left[f\left(\tfrac{1}{2}\right)+f(1)+f(1.5)+f(2)+f(2.5)+f(3)+f(3.5)+f(4)\right]$$

$$= 17.5$$

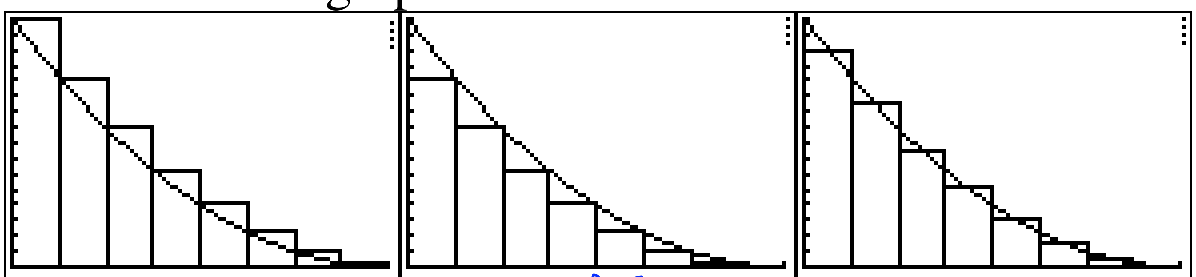
$$\int_0^4 (x-4)^2\,dx \approx \text{MRAM}$$

$$\text{MRAM} = \frac{1}{2}\left[f\left(\tfrac{1}{4}\right)+f\left(\tfrac{3}{4}\right)+f\left(\tfrac{5}{4}\right)+f\left(\tfrac{7}{4}\right)+f\left(\tfrac{9}{4}\right)+f\left(\tfrac{11}{4}\right)+f\left(\tfrac{13}{4}\right)+f\left(\tfrac{15}{4}\right)\right]$$

$$= 21.252$$

Since $\int_0^4 (x-4)^2\,dx = 21\tfrac{1}{3}$, then

LRAM was an __OVER__ - approximation
RRAM was an __UNDER__ - approximation
MRAM was an __UNDER__ - approximation
We should probably remember this for any function that is ***decreasing and concave up***.
Here are some graphs for our Riemann Sums:



LRAM RRAM MRAM

Definition of a Definite Integral

$$\lim_{dx\to 0} \sum_{i=1}^{n} f(c_i)\,dx = \int_a^b f(x)\,dx$$

dx = WIDTH OF RECTANGLE

$f(c_i)$ = heights of Rectangles

What?!!! [Those limits never go away!]
Think of the rectangles! We are summing up the area of numerous rectangles and we want the width of the rectangles $[dx]$ to get close to zero and the number of rectangles, n, to get close to infinity.

♪ If f is ***continuous*** on $[a, b]$, then f can be integrated on $[a, b]$.

For now, we are going to concentrate on finding definite integrals using Riemann sums [one of the RAMs].

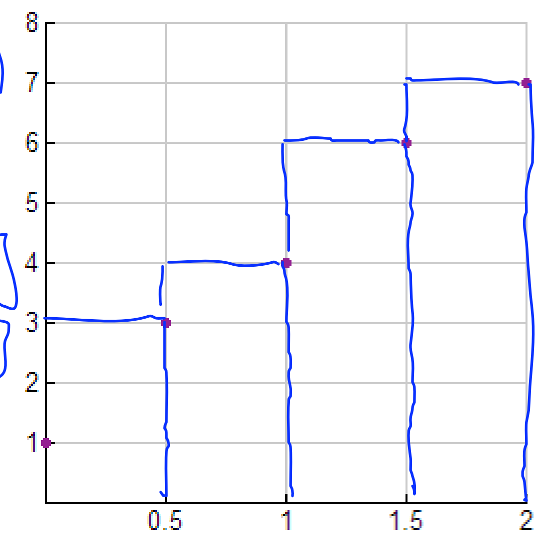
Let's consider the following problem
The table below contains values of a continuous function f at several inputs of x.

x	0	0.5	1	1.5	2
$f(x)$	1	3	4	6	7

Estimate $\int_0^2 f(x)\,dx$ using a right-hand sum [RRAM] with four equal subintervals, and draw a sketch that illustrates this sum geometrically.

$\int_0^2 f(x)dx \approx RRAM$

$RRAM = \frac{1}{2}[f(.5)+f(1)+f(1.5)+f(2)]$
$= \frac{1}{2}[3+4+6+7]$
$= 10$



$$\int_0^2 f(x)\,dx \approx RRAM$$

$$RRAM = 0.5\left[f(0.5) + f(1) + f(1.5) + f(2)\right]$$

$$RRAM = 0.5[3+4+6+7]$$

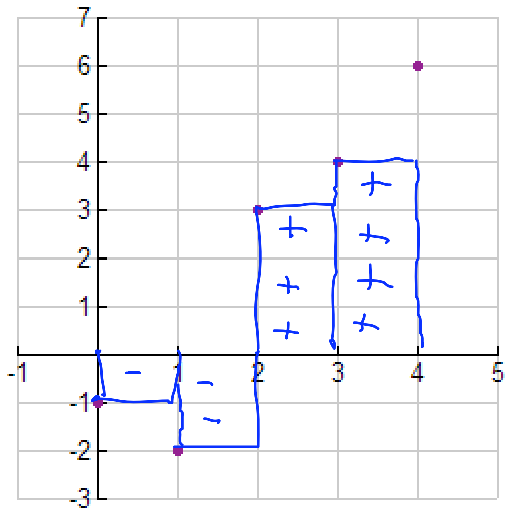
$$RRAM = 0.5[20]$$

$$RRAM = 10$$

We are simply adding up the area of the four rectangles! We need to be careful whenever our $f(x)$ values are negative.

What if we had the following values for $g(x)$ and we want to find $\int_0^4 g(x)\,dx$ using a left-hand sum with four equal subintervals?

x	0	1	2	3	4
$g(x)$	-1	-2	3	4	6



$$\int_0^4 g(x)\, dx \approx \text{LRAM}$$

$$LRAM = (1)\left[g(0) + g(1) + g(2) + g(3)\right]$$
$$LRAM = (1)\left[-1 + -2 + 3 + 4\right]$$
$$LRAM = 4$$

Now let's sing our Riemann Sum Song!

Homework:
Page 279 #45 and 46**- *use these directions*** – find the definite integral using LRAM, RRAM

If we have time:
Let's do some more AP Problems!
The process of estimating definite integrals using the sum of the areas of rectangles is called using a Riemann Sum.

Here is an AP example:

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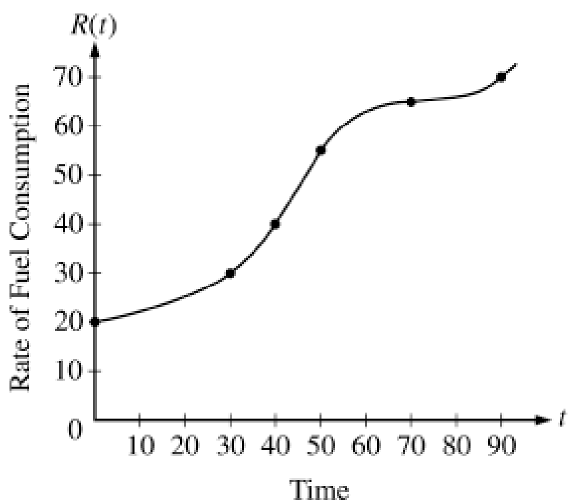

t (minutes)	$R(t)$ (gallons per minute)
0	20
30	30
40	40
50	55
70	65
90	70

3. The rate of fuel consumption, in gallons per minute, recorded during an airplane flight is given by a twice-differentiable and strictly increasing function R of time t. The graph of R and a table of selected values of $R(t)$, for the time interval $0 \le t \le 90$ minutes, are shown above.

(c) Approximate the value of $\int_0^{90} R(t)\,dt$ using a left Riemann sum with the five subintervals indicated by the data in the table. Is this numerical approximation less than the value of $\int_0^{90} R(t)\,dt$? Explain your reasoning.