

Introduction to Integration

Whose derivative is $5x^4$?

If $f'(x) = 5x^4$, then $f(x) = x^5 + C$, where $C \in$ real numbers.

Because: If $f(x) = x^5$, then $f'(x) = 5x^4$

If $f(x) = x^5 + 3$, then $f'(x) = 5x^4$

If $f(x) = x^5 - 4$, then $f'(x) = 5x^4$

Definition of an **anti-derivative**:

A function F is an anti-derivative of f on an interval I if

$$F'(x) = f(x) \text{ for all } x \text{ in } I.$$

Consider $f(x) = 2x$ *$2x$ is some function's derivative*

Let $F(x)$ be an anti-derivative of $f(x)$.

Then $F(x) = x^2 + C$, $C \in$ real numbers

Think of $F(x)$ as the family of functions whose derivative is equal to $2x$

If $f(x) = 2x$ and $f(x)$ is a derivative of $F(x)$, then we can

say that $\frac{dy}{dx} = 2x$

This is called a **differential equation**.

DIFF EQ

Our first differential equation:

$$\frac{dy}{dx} = 5$$

Then $y = 5x + C$

$$\frac{d}{dx} [5x + C] = 5$$

$$\frac{d}{dx} y = \frac{dy}{dx}$$

New notation!!!!

$$\frac{dy}{dx} = f(x)$$

MULTIPLY BOTH SIDES BY dx

$$dy = f(x) dx$$

$$y = \int \underbrace{f(x) dx}_{\text{INTEGRAND}}$$

$$y = F(x) + C$$

$f(x)$ is someone's
DERIVATIVE

\int is an integral sign

$\int f(x) dx$ is called an indefinite integral.

♪ We will also learn definite integrals.

Indefinite integral is a synonym for anti-derivative. Finding the anti-derivative is called anti-differentiating or is called integrating.

In general, for $C \in$ real numbers

$$\int F'(x) dx = F(x) + C$$

OR

$$\int f(x) dx = F(x) + C$$

OR

$$\int f'(x) dx = f(x) + C$$

♪ The solution to our indefinite integral is a family of functions. If you forget your “+C”, then you are a “chucklehead”! **Don't be a chucklehead!** Remember, FHS Calc AB – we're smart and we're pretty! NO chuckleheads here!

See page 250 for Rules of Integration [They are similar to our Rules of Differentiation.]

Derivatives

$$C \in \mathbb{R}$$

$$\frac{d}{dx}[c] = 0$$

$$\frac{d}{dx}[kx] = k$$

$$k \in \mathbb{R}$$

Integrals

$$\int 0 dx = c$$

$$\int k dx = kx + C$$

$$\frac{d}{dx}[k f(x)] = k f'(x)$$

$$\int k f'(x) dx = k f(x) + C$$

$$k \in \mathbb{R}$$

OUR OLD FRIEND CMR

$$\frac{d}{dx}[f(x) \pm g(x)]$$

$$\int [f'(x) \pm g'(x)] dx$$

$$= f'(x) \pm g'(x)$$

$$= f(x) \pm g(x) + C$$

$$\frac{d}{dx} x^n = n x^{n-1}$$

$$\int n x^{n-1} dx = x^n + C$$

Power Rule

Now in action! [How can we check our answer?]

$$\int 7x^6 dx$$

$n=7$

$$= x^7 + C$$

$$\int -2 dx = -2x + C$$

$$\int k dx = kx + C$$

where $k = -2$

$$\int 17 dx = 17x + C$$

$$\text{also } \int k dx \quad k = 17$$

$$\int 5 f'(x) dx = 5 f(x) + C$$

$$\int k f'(x) dx = k f(x) + C$$

$$\int 3x^2 dx = x^3 + C$$

$$\int nx^{n-1} dx = x^n + C$$

$$\int (3x^2 + 17) dx = x^3 + 17x + C$$

$$\int 3x^2 dx + \int 17 dx$$

$$\int 4x^3 dx = x^4 + C$$

$$\int nx^{n-1} dx$$

$n = 4$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad \text{for } n \neq -1$$

$$\int x^7 dx = \frac{x^8}{8} + C$$

$$n = 7$$

$$\frac{d}{dx} \left[\frac{1}{8} x^8 + C \right]$$


$$\int x^{-3} dx = \frac{x^{-2}}{-2} + C$$

$$n = -3$$

Slightly harder:

$$\int 7x^2 dx$$

NOT $\int nx^{n-1} dx$

NOR $\int x^n dx$ 

We can think of this as:

$$7 \int x^2 dx = 7 \left[\frac{x^3}{3} + C \right]$$

CMR
AGAIN
😊

$$= \frac{7x^3}{3} + C$$

Remember, C is a non-specified constant

Some handy tricks:

Rewriting as x^n

Simplifying if it helps

$\int \sqrt[5]{x} dx$ can be rewritten as

$$\int x^{\frac{1}{5}} dx = \frac{x^{\frac{6}{5}}}{\frac{6}{5}} + C$$

OR

$$\frac{5}{6} x^{\frac{6}{5}} + C$$

$\int \frac{1}{x^5} dx$ can be rewritten as

$$\int x^{-5} dx = \frac{x^{-4}}{-4} + C$$

$\int x(x^3 - 3) dx$ can be simplified

$$\int (x^4 - 3x) dx = \frac{x^5}{5} - 3\left(\frac{x^2}{2}\right) + C$$

$\int \left(\frac{1}{7x^3}\right) dx$ can be rewritten as

$$\frac{1}{7} \int x^{-3} dx = \frac{1}{7} \left(\frac{x^{-2}}{-2}\right) + C \quad \text{or}$$

$$\frac{1}{7} \cdot \frac{1}{x^3}$$
$$-\frac{1}{14} x^{-2} + C$$

Note: Do not fall in love with the integral sign!

Try on your own:

Page 255 #15, 17, 21, 23, 27, 29, 33

15. $\int (x+3) dx$
 $= \frac{x^2}{2} + 3x + C$

17. $\int (2x - 3x^2) dx$
 $= x^2 - x^3 + C$

21. $\int (x^{\frac{3}{2}} + 2x + 1) dx$
 $= \frac{2}{5} x^{\frac{5}{2}} + x^2 + x + C$

$$23. \int \sqrt[3]{x^2} dx$$

$$= \int x^{\frac{2}{3}} dx = \frac{3}{5} x^{\frac{5}{3}} + C$$

$$27. \int \frac{x^2 + x + 1}{\sqrt{x}} dx$$

$$= \int \left[x^{\frac{3}{2}} + x^{\frac{1}{2}} + x^{-\frac{1}{2}} \right] dx$$

$$= \frac{2}{5} x^{\frac{5}{2}} + \frac{2}{3} x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$$

$$29. \int (x+1)(3x-2) dx$$

$$= \int [3x^2 + x - 2] dx$$

$$= x^3 + \frac{x^2}{2} - 2x + C$$

$$33. \int dx = x + C \quad \text{think } \int 1 dx$$

More integration! See page 250

$$\int \cos x dx = \sin x + C$$

$$\int \sin x dx = -\cos x + C$$

$$\int -\sin x dx = \cos x$$

$$\int \sec^2 x dx = \tan x + C$$

$$\int \sec x \tan x dx = \sec x + C$$

$$\int \csc^2 x \, dx = -\cot x + C$$

$$\int -\csc^2 x \, dx = \cot x + C$$

$$\int \csc x \cot x \, dx = -\csc x + C$$

$$\int -\csc x \cot x \, dx = \csc x + C$$

The same rules apply for trigonometric integrals – constant multiple rules, sum or difference rule

$$\int 5 \sec^2 x \, dx = \boxed{5} \int \sec^2 x \, dx = 5 \tan x + C$$

CMR

$$\int (3\theta^2 + \cos \theta) \, d\theta$$

$$= 3 \int \theta^2 \, d\theta + \int \cos \theta \, d\theta$$

$$= \theta^3 + \sin \theta + C$$

Knowing your trig really helps!

$$\int \sin \theta (3 + \csc \theta) \, d\theta \quad \text{Why not simplify first?}$$

$$\int (3 \sin \theta + 1) \, d\theta$$

$$= -3 \cos \theta + \theta + C$$

$$\int \overbrace{\csc \theta (\csc \theta - \cot \theta)} d\theta$$

$$= \int [\csc^2 \theta - \csc \theta \cot \theta] d\theta$$

$$= -\cot \theta + \csc \theta + C$$

Here's a good one!

$$\int \frac{\sin \theta}{1 - \sin^2 \theta} d\theta$$

$$= \int \frac{\sin \theta}{\cos^2 \theta} d\theta$$

$$= \int \frac{\sin \theta}{\cos \theta} \cdot \frac{1}{\cos \theta} d\theta$$

$$= \int \tan \theta \sec \theta d\theta = \sec \theta + C$$

More "diff EQ"

Let $\frac{dy}{dx} = 4x$ where my **initial condition** is that $f(0) = 6$

$$\frac{dy}{dx} = 4x$$

$$dy = 4x dx \quad \text{SEPARATE}$$

$$\int dy = \int 4x dx \quad \text{INTEGRATE [BOTH SIDES]}$$

$$\int dy = \int 4x dx$$

$$y + C_1 = 2x^2 + C_2$$

Combine the constants into one constant, call it C and since we are solving for a y , place the constant on the side with x

$$y = 2x^2 + C$$

Now use our initial condition to solve the particular solution [rather than a general solution]

To find C use $f(0) = 6$ in our general solution

$$6 = 0 + C$$

$$6 = C$$

$$\text{let } x = 0 \\ y = 6$$

Hence, $f(x) = 2x^2 + 6$

This is the particular solution that fulfills all of the given information.

Let's try this one:

$\frac{dy}{dx} = 6x^2$ With our initial condition $f(0) = -1$

$$\int dy = 6x^2 dx \\ \int dy = \int 6x^2 dx$$

$$y = 2x^3 + C$$

$$-1 = 0 + C$$

$$y = 2x^3 - 1$$

$$\text{let } x = 0 \\ y = -1$$

$$C = -1$$

Now let's see how we can use this in a physics setting. See page 257 #67

A ball is thrown vertically upward from a height of 6 feet with an initial velocity of 60 feet per second. How high will the ball go?

ON EARTH IN THE USA

Given: $a(t) = -32 \frac{\text{ft}}{\text{sec}^2}$, $v(0) = 60 \frac{\text{ft}}{\text{sec}}$, $s(0) = 6$ feet

INITIAL VELOCITY INITIAL POSITION

We know that $a(t) = v'(t)$ so $v(t) = \int a(t) dt$ ☆☆☆

$$v(t) = \int -32 dt$$
$$v(t) = -32t + C$$

Now we need to use our initial conditions to find C

Since $v(0) = 60$, then $60 = -32(0) + C$

Hence, $C = 60$

So, $v(t) = -32t + 60$

$t = 0$
 $v = 60$

We are still looking for the maximum height so we will eventually need a position function.

Since $v(t) = s'(t)$, then $s(t) = \int v(t) dt$ ☆☆☆☆

$$s(t) = \int (-32t + 60) dt$$

$$s(t) = -16t^2 + 60t + C$$

Once again, use our initial condition to find C.

Since $s(0) = 6$, then $6 = -16(0^2) + 60(0) + C$

Hence, $C = 6$

So, $s(t) = -16t^2 + 60t + 6$

let $t=0$
 $s=6$

OUR POSITION FUNCTION

Now to answer the question. To find maximum height we first need to let $v(t) = 0$

$$0 = -32t + 60$$

$$t = 1.875 \text{ seconds}$$

At $t = 1.875$, $v(t)$ changes from positive to negative values so $t = 1.875$ will give us a relative maximum value.

$$s(1.875) = 62.25 \text{ feet is our maximum height}$$

♪ #68 gives us a general equation for position [in feet]

$$s(t) = -16t^2 + (v_0)t + (s_0)$$

Where $v_0 = v(0) =$ initial velocity

And $s_0 = s(0) =$ initial position

Try #73 on page 257 [Since we are in meters, then

$$a(t) = -9.8 \frac{m}{\text{sec}^2} \text{ [show all integration steps]} \quad v(0) = 10 \frac{m}{\text{sec}}$$

Is there a general equation for meters?

$$s(0) = 2 \text{ m}$$

$$v(t) = \int a(t) dt$$
$$= \int -9.8 dt$$

$$v(t) = -9.8t + C$$

(use $v(0)$)

$$10 = 0 + C \quad \text{Hence } C = 10$$

$$v(t) = -9.8t + 10$$

$$s(t) = \int v(t) dt$$
$$= \int [-9.8t + 10] dt$$

$$s(t) = -4.9t^2 + 10t + C$$

$$2 = 0 + 0 + C$$

$$s(t) = -4.9t^2 + 10t + 2$$

$$s(0) = 2$$

At $t = 1.0204$ $v(t)$ changes
FROM POSITIVE TO NEGATIVE
VALUES Hence the ^{max} will
occur at $t \approx 1.0204$.

max height $\Delta(1.0204) \approx 7.102 \text{ m}$

USA TOSS > NON USA TOSS

OF COURSE

FHS TOSS > BHS TOSS

Homework: page 255 #16 – 34 evens #36, 38, 40, 42 and
page 257 #75, 77, 78 [show all integration steps]

due 11/30

