Separable Differential Equations

$$\frac{dy}{dt} = \cos(2t) \qquad \text{Given: } y(0) = 1$$

$$dy = \cos(2t)\,dt$$

$$\int dy = \int \cos(2t)\,dt$$

$$y = \frac{1}{2}\sin(2t) + C$$

"+ C" must be here!

$$1 = \frac{1}{2}\sin(0) + C$$

hence, C=1

$$y = \frac{1}{2}\sin(2t) + 1$$

Our SOLUTION!!!

$u = 2t$
$du = 2\,dt$
$\frac{1}{2}du = dt$

① SEPARATE ✓
② INTEGRATE ✓
③ SOLVE for C ✓
④ SOLVE for y

The solution to a separable differentiable equation is a differentiable function on an open interval which contains the initial given x- value. [See diffeqspecial.doc]

If no initial value is given, then your solution is the family of functions whose derivative is given as $\frac{dy}{dx}$. [Don't forget the "+C".]

Our four steps:
Separate
Integrate
Solve for C [or solve for y]
Solve for y [or solve for C]

Slightly harder: no initial value given

$$\frac{dy}{dx} = 4 - y$$

Be careful when separating!

$$\frac{dy}{4-y} = dx$$

DIVIDE BOTH SIDES by $(4-y)$ AND MULTIPLY BOTH SIDES by dx

$$\int \frac{dy}{4-y} = \int dx$$

$y < 4$

$$-\ln(4-y) = x + C$$

Goal: Get y by itself!

$$\ln(4-y) = -x + C$$

Remember: C is generic

$$e^{\ln(4-y)} = e^{-x+C}$$

$[e^{-x+C} = \underline{e^{-x} \bullet e^{C}}]$

$$4 - y = Ce^{-x}$$

$$y = 4 - Ce^{-x}$$

Our SOLUTION!

$$y' = \frac{5x}{y}$$

Given: $(0, -1)$ is on the curve

$$\frac{dy}{dx} = \frac{5x}{y}$$

$$y\,dy = 5x\,dx$$

$$\int y\,dy = \int 5x\,dx$$

$$\frac{y^2}{2} = \frac{5x^2}{2} + C$$

$$y^2 = 5x^2 + C$$

Remember – C is generic

$$(-1)^2 = 0 + C$$

Hence, C=1

$$y^2 = 5x^2 + 1$$

Now here is the tricky part: Remember our given point

$$y = \pm\sqrt{5x^2 + 1}$$ Since we must go through (0, -1)

Hence, $y = -\sqrt{5x^2 + 1}$

choose this one

Really tricky! No given initial value

$$\frac{dy}{dx} = xy + 2y$$

We must FACTOR!!!

$$\frac{dy}{dx} = y(x+2)$$

Now we can separate.

Divide both sides by y

multiply both sides by dx

$$\frac{dy}{y} = (x+2)\,dx$$

$$\int \frac{dy}{y} = \int (x+2)\,dx$$

! $y > 0$

$$\ln y = \frac{x^2}{2} + 2x + C$$

$$e^{\ln y} = e^{\left(\frac{x^2}{2}+2x+C\right)} = e^{\frac{x^2}{2}+2x} \cdot e^{C}$$ let $C = e^{C}$

$$y = Ce^{\left(\frac{x^2}{2}+2x\right)}$$

You try:

$$\frac{dy}{dx} = \frac{x^3}{y^2}$$

Given: $y(2) = 3$

"STEP BY STEP"

1. SEPARATE
2. INTEGRATE
3. SOLVE FOR C
4. SOLVE FOR y

$$y^2\,dy = x^3\,dx$$

$$\int y^2\,dy = \int x^3\,dx$$

$$\frac{y^3}{3} = \frac{x^4}{4} + C$$

$$\frac{3^3}{3} = \frac{2^4}{4} + C$$

Hence $C = 5$

$$\frac{y^3}{3} = \frac{x^4}{4} + 5$$

$$y^3 = \frac{3x^4}{4} + 15$$

$$y = \sqrt[3]{\frac{3x^4}{4} + 15}$$

Here's a tricky one to try:

$$\frac{dy}{dx} = \frac{3x}{y}$$

Given: $y(6) = -4$

$y\,dy = 3x\,dx$

$\int y\,dy = \int 3x\,dx$

$\frac{y^2}{2} = \frac{3x^2}{2} + C$

$\frac{(-4)^2}{2} = \frac{3(6^2)}{2} + C$

$\therefore C = -46$

$\frac{y^2}{2} = \frac{3x^2}{2} - 46$

$y^2 = 3x^2 - 92$

$y = \pm\sqrt{3x^2 - 92}$

consider given point $(6, -4)$

$y = -\sqrt{3x^2 - 92}$

If time, try:

$$\frac{dy}{dx} = x^2\sqrt{y} \qquad \text{Given: } y(0) = 9$$

$$\frac{dy}{\sqrt{y}} = x^2\,dx$$

$$\int \underline{\underline{y^{-\frac{1}{2}}}}\,dy = \int x^2\,dx$$

$$2\sqrt{y} = \frac{x^3}{3} + C$$

$$2\sqrt{9} = 0 + C \qquad \therefore C = 6$$

$$2\sqrt{y} = \frac{x^3}{3} + 6$$

$$\sqrt{y} = \frac{x^3}{6} + 3$$

$$y = \left(\frac{x^3}{6} + 3\right)^2$$

Common Applications:

Let $s(t)$ be the position function
We already know that

$$\frac{ds}{dt} = v(t)$$

$$ds = v(t)\ dt$$

$$\int ds = \int v(t)\ dt$$

$$s(t) = \int v(t)\, dt$$

Don't forget that you would need a "+C"

Likewise:

$$v(t) = \int a(t)\, dt$$

$$\frac{dV}{dt} = a(t)$$

$$\int dV = \int a(t)\, dt$$

Reminder:

Speed = $|v(t)|$

Total Distance Traveled on $[a, b] = \int_a^b |v(t)|\, dt$

Displacement on $[a,b] = \int_a^b v(t)\, dt$

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