♫ CHAIN RULE SONG ♫

[Sung to the tune of *America the Beautiful,* words by Ms. McCleary]

The Chain Rule is not hard to do
We need to sub a "u"
The "u" is just the inside of
Composite "f of u"
We'll multiply f prime of u
Times du/dx , Hence,
dx of f of g of x is
Found with diligence

Look for patterns:

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\sin(2x)$ $\quad u=2x$ $\quad \frac{du}{dx}=2$	$2\cos(2x)$
$\sin(3x)$	$3\cos(3x)$
$\sin(4x)$	$4\cos(4x)$
$\sin(nx)$ $\sin(u)$	$n\cos(nx)$ $\cos u \frac{du}{dx}$

How about:

$$\frac{d}{dx}\cos u = -\sin u \frac{du}{dx}$$

$$\frac{d}{dx}\tan u = \sec^2 u \frac{du}{dx}$$

Now let's look at some more patterns:

y	y'
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
$\sqrt{3x}$ $u=3x$ $\frac{du}{dx}=3$	$\frac{3}{2\sqrt{3x}}$
$\sqrt{7x}$	$\frac{7}{2\sqrt{7x}}$
$\sqrt{11x}$	$\frac{11}{2\sqrt{11x}}$
$\sqrt{g(x)}$	$\frac{g'(x)}{2\sqrt{g(x)}}$
$\sqrt{g(x)+h(x)}$	$\frac{g'(x)+h'(x)}{2\sqrt{g(x)+h(x)}}$
$y=\sqrt{u}$ where u is a differentiable function of x	$\frac{\frac{du}{dx}}{2\sqrt{u}}$

$g(x)$	$g'(x)$
$\frac{1}{x}$	$-\frac{1}{x^2}$
$\frac{1}{2x}$ $u = 2x$ $\frac{du}{dx} = 2$	$\frac{-2}{(2x)^2}$
$\frac{1}{3x}$	$\frac{-3}{(3x)^2}$
$\frac{1}{f(x)}$	$\frac{-f'(x)}{f^2(x)}$
$\frac{1}{f(x)+h(x)}$	$\frac{-[f'(x)+h'(x)]}{[f(x)+h(x)]^2}$
$\frac{1}{u}$ where u is a differentiable function of x	$\frac{-\frac{du}{dx}}{u^2}$

More of the Chain Rule

Sometimes you have to use the Chain Rule more than once. For instance, let $y = \sin^2(3x)$

We could think of this as: $y = (\sin(3x))^2$

$y = [\underbrace{\sin(3x)}_{u}]^2$

If we did, then $y = u^2$ where $u = \sin(3x)$

If $u = \sin(3x)$, then $\frac{du}{dx}$ would need to be found using the Chain Rule. Good thing we already know this.

If $u = \sin(3x)$, then $\frac{du}{dx} = 3\cos(3x)$

Now we can find y' [We can do this!!!]

$y = (\sin(3x))^2$

$y = u^2$

$y' = 2u \bullet \frac{du}{dx}$

CHAIN RULE

$u = \sin(3x)$

$\frac{du}{dx} = 3\cos(3x)$

$y' = 2\underbrace{\sin(3x)}_{u} \bullet \underbrace{3\cos(3x)}_{\frac{du}{dx}}$

$y' = 6\sin(3x)\cos(3x)$

Now let us consider $y = \sqrt{\tan(4x)}$

Stupid @#$%! $\sqrt{}$ Let us rewrite as $y = (\tan(4x))^{\frac{1}{2}}$

$y = [\tan(4x)]^{\frac{1}{2}}$

We could think of this as $y = u^{\frac{1}{2}}$ where $u = \tan(4x)$

Then $\dfrac{du}{dx} = 4\sec^2(4x)$. We are ready to find y'

$u = \tan(4x)$

$\dfrac{du}{dx} = 4\sec^2(4x)$

$y = (\tan(4x))^{\frac{1}{2}}$

Think $y = u^{\frac{1}{2}}$

Then $y' = \dfrac{1}{2\sqrt{u}} \bullet \dfrac{du}{dx}$

$$y' = \frac{1}{2\sqrt{\tan(4x)}} \bullet \overbrace{4\sec^2(4x)}^{\frac{du}{dx}}$$

$$y' = \frac{2\sec^2(4x)}{\sqrt{\tan(4x)}}$$

♪ This does not need to rationalized for the AP people. So, I won't bother.

==Beware of poor reading skills==! [When would we need the Chain Rule to find y'?]

$y=\cos 3x^2$ is read as $y=\cos\left(3x^2\right)$

$u=3x^2$

$y'=-6x\sin(3x^2)$

$\frac{du}{dx}=6x$

$y=(\cos 3)x^2$ is read as $y=\underbrace{(\cos 3)}_{\text{constant}}\bullet x^2$ where cos3 is a constant

$y'=(2\cos 3)x$

No chain rule needed

$y=\cos(3x)^2$ is read as $y=\cos\left(9x^2\right)$

$u=9x^2$

$y'=-18x\sin(9x^2)$

$\frac{du}{dx}=18x$

$y=\cos^2 x$ is read as $y=(\cos x)^2$

$y=u^2$

$u=\cos x$

$y'=-2\cos x\sin x$

$\frac{du}{dx}=-\sin x$

$y=\cos^2\left(3x^2\right)$ is read as $y=\left[\cos\left(3x^2\right)\right]^2$

$y=u^2$

$y'=2u\frac{du}{dx}$

$y'=-12x\cos(3x^2)\sin(3x^2)$

Need the chain rule twice!!

$u=\cos(3x^2)$

$\frac{du}{dx}=-6x\sin(3x^2)$

$y=\sqrt{\cos x}$ can be read as $y=(\cos x)^{\frac{1}{2}}$

$u=\cos x$

$y'=\frac{-\sin x}{2\sqrt{\cos x}}$

$\frac{du}{dx}=-\sin x$

Try:

Let $f(x) = \sin(\sqrt{x})$. Find $f'(x)$

$u = \sqrt{x}$

$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$

$f(x) = \sin u$

$f'(x) = \cos u \frac{du}{dx}$

$f'(x) = \frac{\cos\sqrt{x}}{2\sqrt{x}}$

Let $g(x) = \tan^2(x^2)$. Find $g'(x)$

$u = \tan(x^2)$

$\frac{du}{dx} = 2x\sec^2 x^2$

$g(x) = [\tan(x^2)]^2 = u^2$

$g'(x) = 2u \frac{du}{dx}$

$g'(x) = 4x\tan(x^2)\sec^2(x^2)$

Let $h(x) = \sqrt{\sin(2x)}$. Find $h'(x)$

$\boxed{u = \sin(2x) \quad \frac{du}{dx} = 2\cos 2x}$

$h(x) = (\sin(2x))^{\frac{1}{2}}$

$h(x) = u^{\frac{1}{2}}$

$h'(x) = \frac{1}{2\sqrt{u}} \frac{du}{dx}$

$= \frac{\cancel{2}\cos(2x)}{\cancel{2}\sqrt{\sin(2x)}} = \frac{\cos(2x)}{\sqrt{\sin(2x)}}$

Now that we know how to use the Chain Rule, we can do some of the problems from our chapter's handouts.

Finding Numerical Derivatives [where a graphing calculator won't help you]

Suppose that the function $f(x)$ and its first derivative have the following values at $x=0$ and $x=1$

x	$f(x)$	$f'(x)$
0	9	-2
1	-3	$\frac{1}{5}$

Find the first derivatives of the following combinations at the given values of x.

(a) $\sqrt{x}\, f(x)$ at x = 1

(b) $\sqrt{f(x)}$ at x = 0

(c) $\dfrac{f(x)}{1+\cos x}$ at x = 0

(d) $f^2(x)$ at x = 0

(e) $\dfrac{1}{f(x)}$ at x = 1

Do (b) and (e)

(b) $\dfrac{d}{dx}\sqrt{f(x)}$ at $x=0$

$$\boxed{\begin{array}{l} u=f(x) \\ \dfrac{du}{dx}=f'(x) \end{array}}$$

$$= \frac{\frac{du}{dx}}{2\sqrt{u}}$$

$$= \left.\frac{f'(x)}{2\sqrt{f(x)}}\right|_{x=0}$$

$$= \frac{f'(0)}{2\sqrt{f(0)}} = \frac{-2}{2\sqrt{9}} = -\frac{1}{3}$$

(e) $\dfrac{d}{dx}\dfrac{1}{f(x)}$ at $x=1$

$$= \frac{d}{dx}\left[f(x)\right]^{-1}$$

$$= -\frac{\frac{du}{dx}}{u^2}$$

$$= \left.\frac{-f'(x)}{f^2(x)}\right|_{x=1} = \frac{-f'(1)}{[f(1)]^2} = \frac{-\frac{1}{5}}{9} = -\frac{1}{45}$$

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	8	2	5	$\frac{1}{3}$
3	3	-4	$-\frac{1}{3}$	$-\frac{8}{3}$

On a separate sheet of paper, find the following:

(a) $\frac{d}{dx}[2f(x)]$ at $x=2$

(b) $\frac{d}{dx}[f(x)+g(x)]$ at $x=3$

(c) $\frac{d}{dx}[f(x)\bullet g(x)]$ at $x=3$

(d) $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]$ at $x=2$

(e) $\frac{d}{dx}[f(g(x))]$ at $x=2$

(f) $\frac{d}{dx}\left[\sqrt{f(x)}\right]$ at $x=2$

(g) $\frac{d}{dx}\left[\frac{1}{g(x)}\right]$ at $x=3$

(e) $\frac{d}{dx}[f(g(x))]$ at $x=2$

$u=g(x)$

$\frac{du}{dx}=g'(x)$

$= f'(u)\frac{du}{dx}$

$= f'(g(x))\,g'(x)$

at $x=2$

$f'(g(2))\,g'(2)$

$(5)\left(\frac{1}{3}\right)=\frac{5}{3}$

Now we can do (e) and (f)!

(f) $\frac{d}{dx}\sqrt{f(x)}$ at $x=2$

$= \left.\frac{f'(x)}{2\sqrt{f(x)}}\right|_{x=2}$

$= \frac{f'(2)}{2\sqrt{f(2)}}$

$= \frac{5}{2\sqrt{8}} = \frac{5}{4\sqrt{2}}$

If you need more practice, then check out the “weird” handout

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	−3	4	5	−1
2	5	1	3	7
3	−2	2	1	8
4	6	5	2	−3
5	9	3	4	−4

a. $H(x)=5f(x)-3g(x)$. Find $H'(1)$

b. $H(x)=\dfrac{f(x)}{g(x)}$. Find $H'(4)$

c. $H(x)=[g(x)]^5$. Find $H'(2)$

d. $H(x)=g(x^2-7)$. Find $H'(3)$

e. $H(x)=f(g(x))$. Find $H'(5)$

EXTRA HOMEWORK CREDIT

Homework: Read 2.4, do page 137 # 25, 26, 27 41, 43, 45, 47, 51, 53 [Please state your u and $\dfrac{du}{dx}$ and clearly show your steps using standard mathematical notation, blah, blah, blah, …]

Note: Some problems requires the Chain Rule AND the Product Rule, or they require the Chain Rule AND the Quotient Rule