Our BIG Free Response Project for 1st Semester:

Part I – Non-calculator Problems

1. On the axes below, draw a function which has all of the following characteristics:

(a) $\lim_{x\to 3^-} f(x) = \infty$

(b) $\lim_{x\to 3^+} f(x) = \infty$

VERTICAL ASYMPTOTE

(c) $\lim_{x\to \infty} f(x) = 8$

(d) $\lim_{x\to -\infty} f(x) = -8$

horizontal asymptotes

(e) $\lim_{x\to 0} f(x) = 1$ REMOVABLE DISCONTINUITY

(f) $f(0) = 0$ POINT

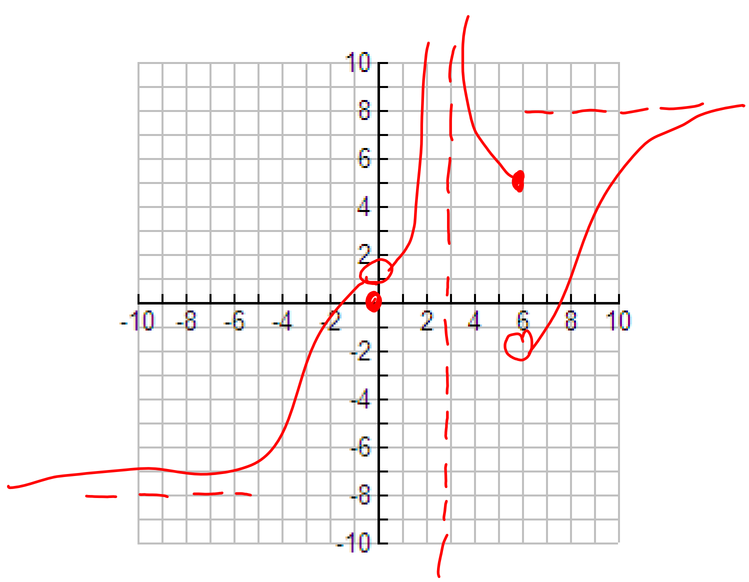
(g) $\lim_{x\to 6^-} f(x) = 5$

(h) $\lim_{x\to 6^+} f(x) = -2$

JUMP DISCONTINUITY

(i) $f(6) = 5$ Point

Something like this



2. Let f be a twice-differentiable function such that $f(2)=5$ and $f(5)=2$. Let $h(x)=f(x)-x$. Explain why there must be a value r for $2 < r < 5$ such that $h(r)=0$.

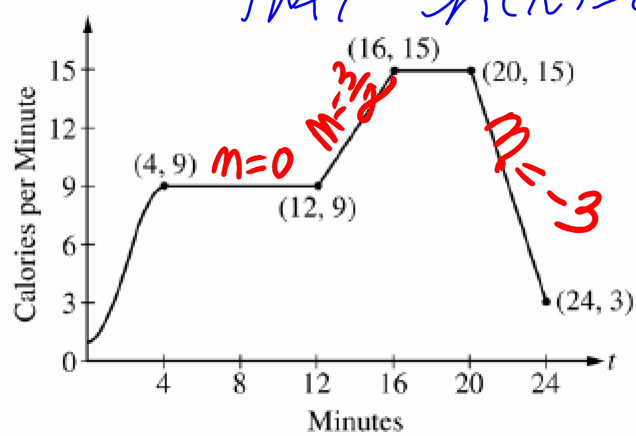
$h(2) = f(2) - 2 = 3$

$h(5) = f(5) - 5 = -3$

By IVT, there is an r, $2 < r < 5$, such that $-3 < h(r) < 3$. Hence there must exist an r, $2 < r < 5$, such that $h(r) = 0$

$h(5) < h(r) < h(2)$

3.



The rate, in calories per minute, at which a person using an exercise machine burns calories is modeled by the function f. In the figure above, $f(t) = -\frac{1}{4}t^3 + \frac{3}{2}t^2 + 1$ for $0 \le t \le 4$ and f is piecewise linear for $4 \le t \le 24$.

(a) Find $f'(22)$. Indicate units of measure.

$$f'(22) = \frac{f(24)-f(20)}{24-20} = \frac{15-3}{20-24} = -3 \ \frac{\text{CALORIES}}{\text{MIN}^2}$$

(b) For the time interval $0 \le t \le 24$, at what time t is f increasing at its greatest rate? Show the reasoning that supports your answer.

f is increasing on $(0,4)$ and $(12,16)$

$f'(t) = \frac{3}{2}$ on $(12,16)$

on $(0,4)$ $f'(t) = -\frac{3}{4}t^2 + 3t$

$f''(t) = -\frac{3}{2}t + 3$ ($f'(t)$'s rate of Δ)

$0 = -\frac{3}{2}t + 3 \qquad t = 2$

At $t = 2$ $f''(t)$ changes from positive to negative values. Hence $f'(t)$ has a rel max at $t = 2$. The rel max value is $f'(2) = 3$. Since $3 > \frac{3}{2}$ then f is increasing at its greatest rate at $t = 2$

t (sec)	0	15	25	30	35	50	60
$v(t)$ (ft/sec)	−20	−30	−20	−14	−10	0	10
$a(t)$ (ft/sec^2)	1	5	2	1	2	4	2

A car travels on a straight track. During the time interval $0 \le t \le 60$ seconds, the car's velocity v, measured in feet per second, and acceleration a, measured in feet per second per second, are continuous functions. The table above shows selected values of these functions.

(a) For $0 < t < 60$, must there be a time t when $v(t) = -5$? Justify your answer.

ONE WAY

By IVT, there is a t, $35 < t < 50$ which is contained in $0 < t < 60$, such that $-10 < v(t) < 0$. Hence there must be a t, $0 < t < 60$, such that $v(t) = -5$

(b) For $0 < t < 60$, must there be a time t when $a(t) = 0$? Justify your answer.

By MVT, there is a t, $0 < t < 25$, which is contained in $0 < t < 60$, such that $v'(t) = a(t) = \frac{v(25) - v(0)}{25 - 0} = 0$ Hence, there is a time, $0 < t < 60$ such that $a(t) = v'(t) = 0$

5. Consider the curve given by $y^2 = 2 + xy$.

(a) Show that $\frac{dy}{dx} = \frac{y}{2y - x}$.

$$\frac{d}{dx} y^2 = \frac{d}{dx} 2 + \frac{d}{dx} \underline{x}\,\underline{y}$$

$$2y \frac{dy}{dx} = 0 + y + x \frac{dy}{dx}$$

$$\frac{dy}{dx}(2y - x) = y$$

$$\frac{dy}{dx} = \frac{y}{2y - x}$$

yay!

(b) Find all points (x, y) on the curve where the line tangent to the curve has slope $\frac{1}{2}$.

let $\frac{dy}{dx} = \frac{1}{2}$

$$\frac{1}{2} = \frac{y}{2y - x}$$

$$2y - x = 2y$$

$$x = 0$$

let $x = 0$ in $y^2 = 2 + xy$

$$y^2 = 2 + 0$$

$$y = \pm\sqrt{2}$$

Hence, Points where $\frac{dy}{dx} = \frac{1}{2}$ are $(0, \sqrt{2})$ and $(0, -\sqrt{2})$

(c) Show that there are no points (x, y) on the curve where the line tangent to the curve is horizontal.

let $\frac{dy}{dx} = 0$

$$0 = \frac{y}{2y - x}$$

so y WOULD need TO equAL 0

let $y=0$ in curve

$y^2 = 2 + xy$

$0 \neq 2 + 0$

Since $0 \neq 2$, then the curve has no horizontal tangents.

6. [calculator-friendly]

t (minutes)	0	2	5	7	11	12
$r'(t)$ (feet per minute)	5.7	4.0	2.0	1.2	0.6	0.5

The volume of a spherical hot air balloon expands as the air inside the balloon is heated. The radius of the balloon, in feet, is modeled by a twice-differentiable function r of time t, where t is measured in minutes. For $0 < t < 12$, the graph of r is concave down. The table above gives selected values of the rate of change, $r'(t)$, of the radius of the balloon over the time interval $0 \le t \le 12$. The radius of the balloon is 30 feet when $t = 5$. (Note: The volume of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$.)

(a) Estimate the radius of the balloon when $t = 5.4$ using the tangent line approximation at $t = 5$. Is your estimate greater than or less than the true value? Give a reason for your answer.

Point $(5, 30)$ $\quad m_{TAN}\big|_{t=5} = 2$

equation of tangent line at $t=5$:

$r - 30 = 2(t-5)$

estimate with $t = 5.4$

$r - 30 = 2(5.4-5)$

$r = 30.8$ ft

Since the graph of $r(t)$ is concave down then this estimate is greater than the actual $r(5.4)$

(b) Find the rate of change of the volume of the balloon with respect to time when $t = 5$. Indicate units of measure.

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

at $t=5$: $\frac{dV}{dt} = 4\pi(30\,ft)^2\left(2\,\frac{ft}{min}\right)$

$$\left.\frac{dV}{dt}\right|_{t=5} = 7200\pi\,\frac{ft^3}{min}$$

(c) Use a right Riemann sum with the five subintervals indicated by the data in the table to approximate $\int_0^{12} r'(t)\,dt$. Using correct units, explain the meaning of $\int_0^{12} r'(t)\,dt$ in terms of the radius of the balloon.

$$\int_0^{12} r'(t)\,dt \approx \text{RRAM}$$

$$\begin{aligned}\text{RRAM} &= 2r'(2) + 3r'(5) + 2r'(7) + 4r'(11) + r'(12)\\ &= 2(4) + 3(2) + 2(1.2) + 4(.6) + .5\\ &= 19.3\ ft\end{aligned}$$

$\int_0^{12} r'(t)\,dt$ is the change in the radius in feet from $t = 0$ to $t = 12$ minutes